# Stationary Phase Monte Carlo Methods

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## Introduction:

- Motivations and Outline of the Method;
  - phase transitions,
  - smoothing of the action,
- Theory Behind;
  - mollification/approximate identity method,
- Examples and Results;
  - Airy function action,
  - $\phi^4$ action,
- Conclusion.

## Outline:

The basic idea is to cast the partition function in such a way that its computation is manageable. Thus, the calculation of its derivatives (which yield the different Green's functions) become a feasible task, as well.

In order to do so, the full [complex] exponential weight is convoluted with a "probability-type" function, so that its oscillations are smoothed out and a Monte Carlo calculation can be performed.

This is particularly interesting when dealing with phase transitions since the factor $\exp(i\,S(x))$ oscillates rapidly and these can be tamed — via the smoothing procedure that will be described shortly — so that a Monte Carlo computation can be performed. Also, coalescing stationary points are better handled by this method, again because of this smoothing technique.

**Theory:**

The whole business is built upon the following construction:

$$\int f(x)\,dx \mapsto \int \langle f(x)\rangle_\epsilon\,dx \; , \tag{1}$$

where

$$\begin{aligned} \langle f(x)\rangle_\epsilon &\equiv (f * P_\epsilon)(x) \; , \\ &= \int P_\epsilon(x-y)\,f(y)\,dy \; , \end{aligned} \tag{2}$$

with

$$\int P_\epsilon(x)\,dx = 1 \; .$$

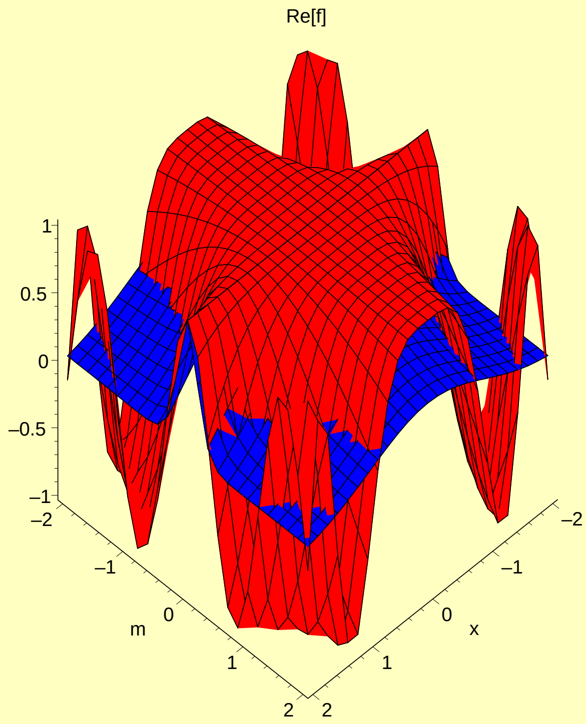
In this case, the smoothing function is given by $P_\epsilon(x)$ which is called a *"mollifier"* or an *"approximate identity"*. The nice property that they have is:

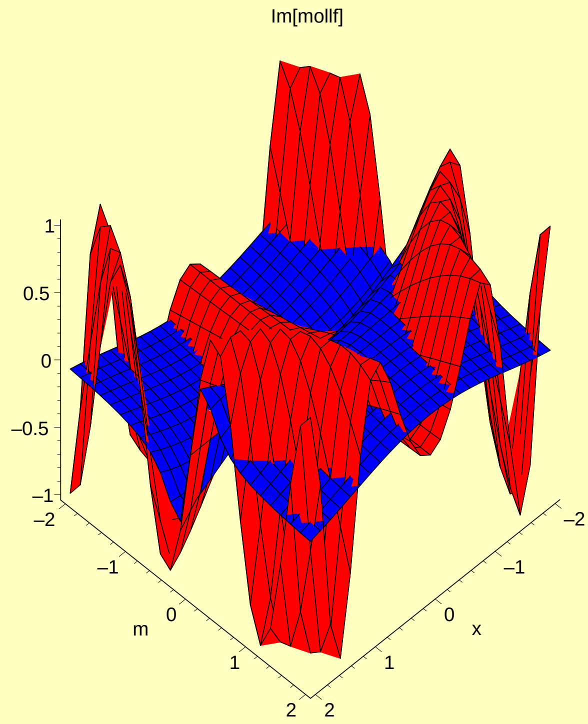
$$\langle f(x)\rangle_\epsilon \xrightarrow{\epsilon\to 0} f(x) \;\Rightarrow\; \int \langle f(x)\rangle_\epsilon\,dx \xrightarrow{\epsilon\to 0} \int f(x)\,dx \; .$$

## Simple example:

$$f_m(x) = \exp(i\,m\,x^2)\ , \qquad P_\epsilon(x) = \frac{\exp\left\{-\frac{1}{2}\,\frac{x^2}{\epsilon^2}\right\}}{\sqrt{2\,\pi\,\epsilon^2}}$$

$$\Rightarrow \qquad \langle f_m(x)\rangle_\epsilon = \frac{\exp\left\{\frac{i\,m\,x^2}{1-2\,i\,m\,\epsilon^2}\right\}}{\sqrt{1-2\,i\,m\,\epsilon^2}}$$



The method's application is in two steps:

1. Pre-averaging of the integrand: the generating functional is given by,

$$\mathcal{Z}_\epsilon[j] = \frac{\int \left\langle e^{i\,S(\mathbf{x})-i\,\mathbf{j}\cdot\mathbf{x}} \right\rangle_\epsilon [dx]}{\int \left\langle e^{i\,S(\mathbf{x})} \right\rangle_\epsilon [dx]} \ .$$

Thus, performing a *saddle-point expansion*:

$$P_\epsilon(\mathbf{y}) = \frac{\exp\left\{-\frac{1}{2}\,\mathbf{y}^\mathsf{T}\cdot(\boldsymbol{\epsilon}^2)^{-1}\cdot\mathbf{y}\right\}}{\sqrt{(2\pi)^n\,\det(\boldsymbol{\epsilon}^2)}}\,,$$

$$\mathcal{I}_\epsilon = \int \left\langle f(\mathbf{x})\,e^{i\,S(\mathbf{x})-i\,\mathbf{j}\cdot\mathbf{x}} \right\rangle_\epsilon [dx]$$

$$\approx f(\mathbf{x_0})\,e^{i\,S(\mathbf{x_0})-i\,\mathbf{j}\cdot\mathbf{x_0}}\times \qquad (3)$$
$$\times \int \frac{\exp\left\{-\frac{1}{2}\,\mathbb{B}^\mathsf{T}\cdot(\mathbb{1}+\boldsymbol{\epsilon}^\mathsf{T}\cdot\mathbb{S}''\cdot\boldsymbol{\epsilon})^{-1}\cdot\mathbb{B}\right\}}{\sqrt{\det(\mathbb{1}+\boldsymbol{\epsilon}^\mathsf{T}\cdot\mathbb{S}''\cdot\boldsymbol{\epsilon})}}\,[dx]\,,$$

where $\boldsymbol{\epsilon}^2$ is the covariance matrix that defines the Gaussian distribution, $\mathbb{B} = \boldsymbol{\epsilon}\cdot\boldsymbol{\nabla}S(\mathbf{x})$ and $[\mathbb{S}'']_{ij} = \partial^2 S(\mathbf{x})/\partial x_i\,\partial x_j$ (Hessian matrix for $S$).

2. Importance Sampling (MC): better handle the MC simulations. The equations become:

$$\mathcal{Z}_\epsilon[j] = \frac{\int \left[ \frac{\left\langle e^{i\,S(\mathbf{x}) - i\,\mathbf{j}\cdot\mathbf{x}} \right\rangle_\epsilon}{W(\mathbf{x})} \right] [dW(\mathbf{x})]}{\int \left[ \frac{\left\langle e^{i\,S(\mathbf{x})} \right\rangle_\epsilon}{W(\mathbf{x})} \right] [dW(\mathbf{x})]} \ , \qquad (4)$$

where $W(\mathbf{x})$ is known as the *importance function* and the MC (Metropolis) sampling is done over this new measure. A reasonable choice for it is given by,

$$W(\mathbf{x}) = W_\epsilon(\mathbf{x}) = \left| \left\langle e^{i\,S(\mathbf{x})} \right\rangle_\epsilon \right| \ .$$

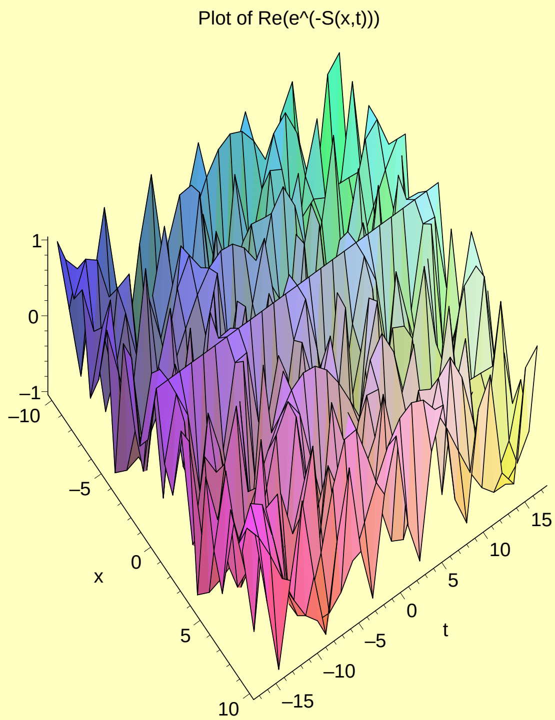
And, just as done before (saddle-point expansion), this becomes:

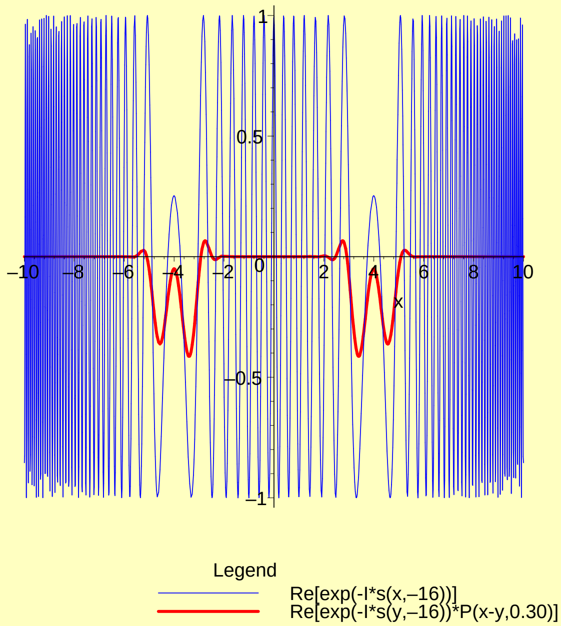
$$W_\epsilon(\mathbf{x}) = \left| \frac{\exp\left\{ i\,S(\mathbf{x_0}) - \frac{1}{2}\,\mathbb{B}^\mathsf{T} \cdot (\mathbb{1} + \boldsymbol{\epsilon}^\mathsf{T} \cdot \mathbb{S}'' \cdot \boldsymbol{\epsilon})^{-1} \cdot \mathbb{B} \right\}}{\sqrt{\det(\mathbb{1} + \boldsymbol{\epsilon}^\mathsf{T} \cdot \mathbb{S}'' \cdot \boldsymbol{\epsilon})}} \right|$$

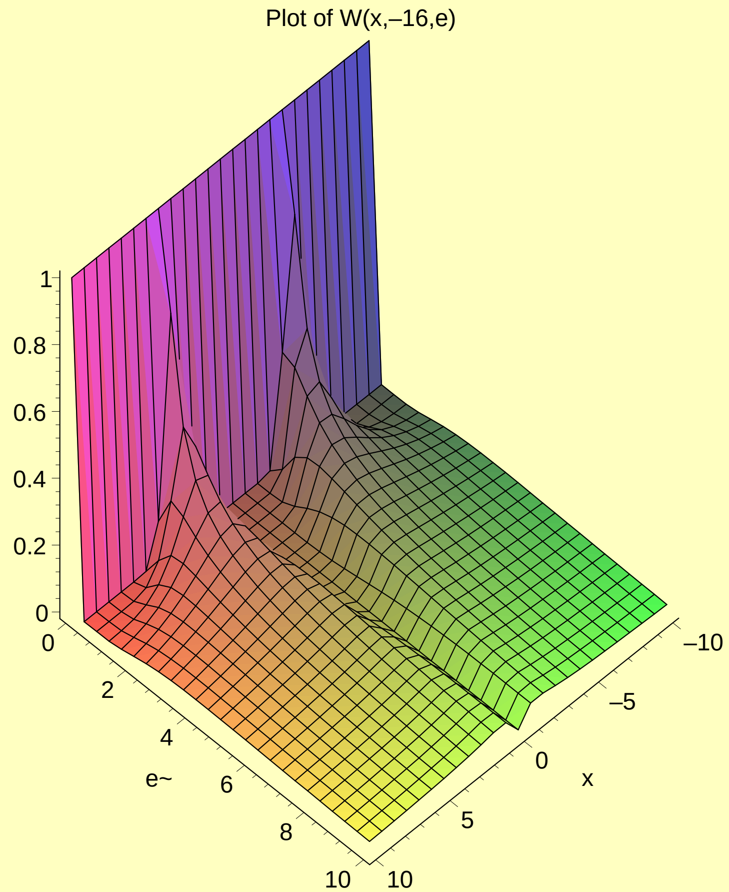
## Example: Airy Function

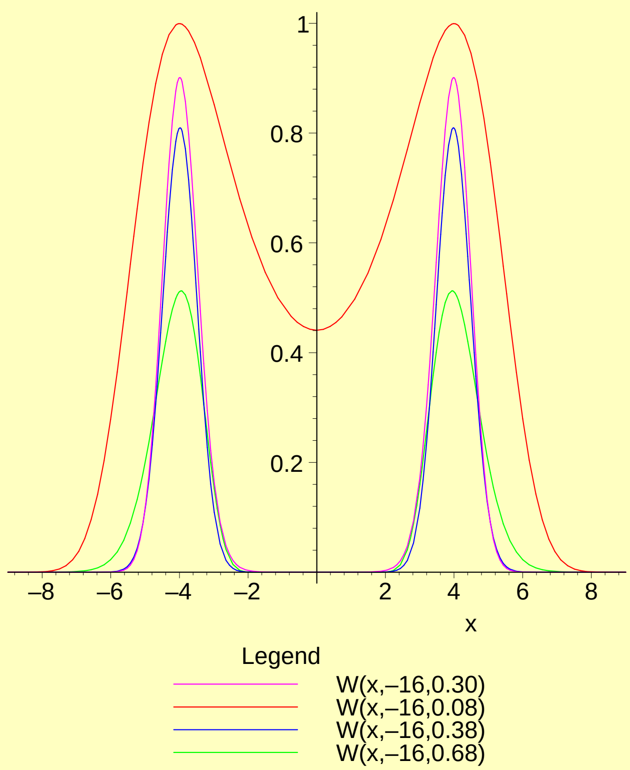
The action is: $S(x) = x^3/3 + t\,x$. Thus,

$$\mathcal{Z}[t] = \frac{\int_{-\infty}^{\infty} \exp\left\{ i\,\frac{x^3}{3} + i\,t\,x \right\} dx}{\int_{-\infty}^{\infty} \exp\left\{ i\,\frac{x^3}{3} \right\} dx} \equiv \frac{\mathrm{Ai}(t)}{\mathrm{Ai}(0)}$$



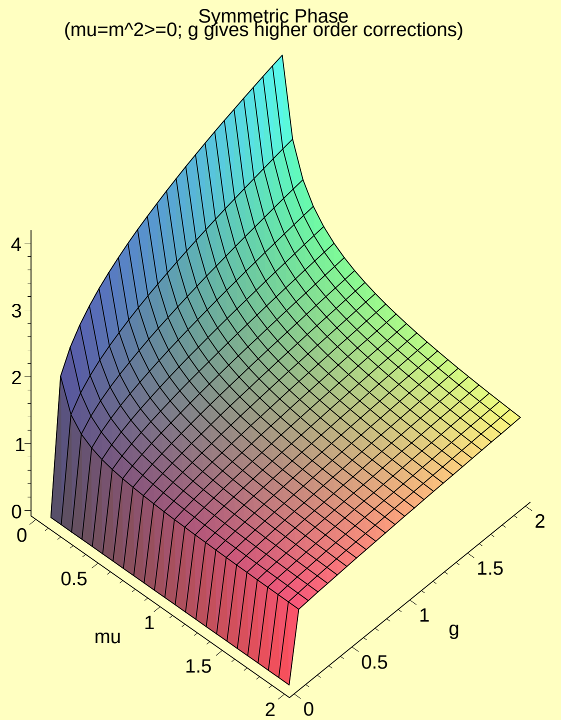

Plot of W(x,–16,e)
1
0.8
0.6
0.4
0.2
0
0
2
4
e~
6
8
10
10
5
0
x
–5
–10

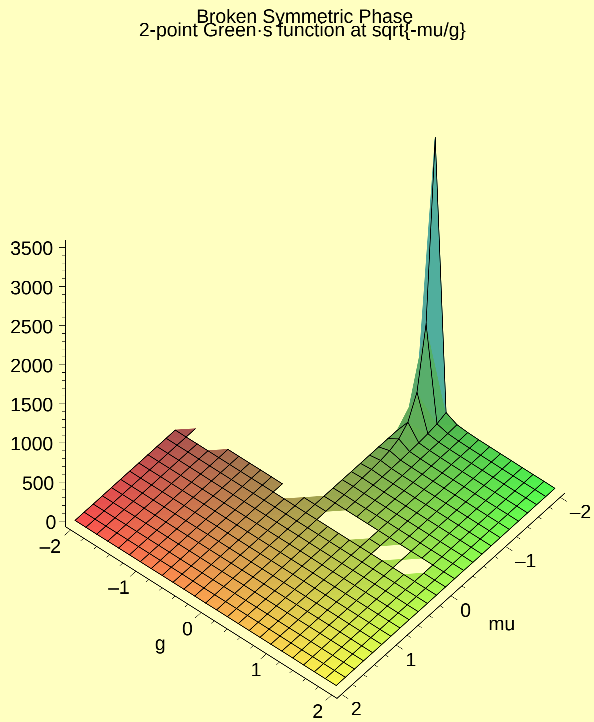

1
0.8
0.6
0.4
0.2
–8
–6
–4
–2
2
4
6
8
x
Legend
W(x,–16,0.30)
W(x,–16,0.08)
W(x,–16,0.38)
W(x,–16,0.68)

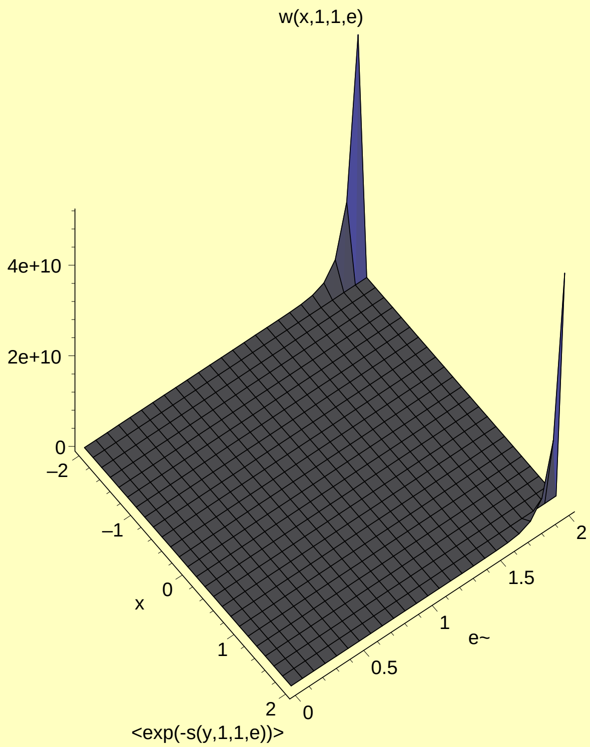
## Example: 0-dimensional $\phi^4$ Theory

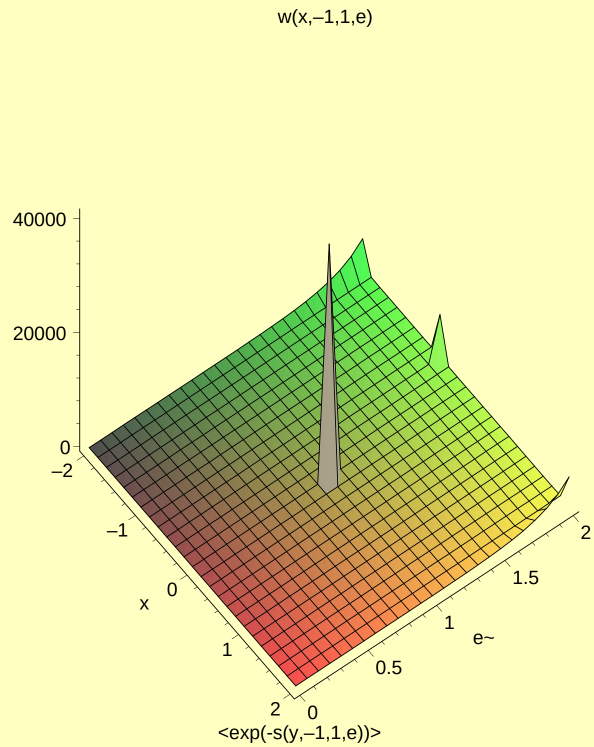
The action is: $S(\phi) = \mu\,\phi^2/2 + g\,\phi^4/4$. Thus,

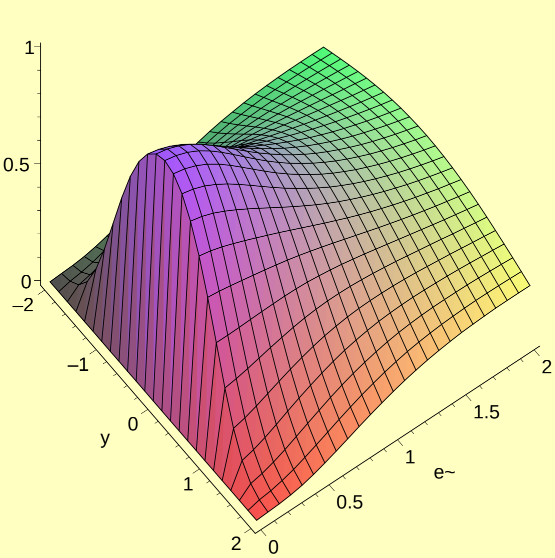
$$\mathcal{Z}_\epsilon[j] = \frac{\int\limits_{-\infty}^{\infty} \left\langle \exp\left\{ i\,\frac{\mu}{2}\,\phi^2 + i\,\frac{g}{4}\,\phi^4 - i\,j\,\phi \right\} \right\rangle_\epsilon d\phi}{\int\limits_{-\infty}^{\infty} \left\langle \exp\left\{ i\,\frac{\mu}{2}\,\phi^2 + i\,\frac{g}{4}\,\phi^4 \right\} \right\rangle_\epsilon d\phi}$$

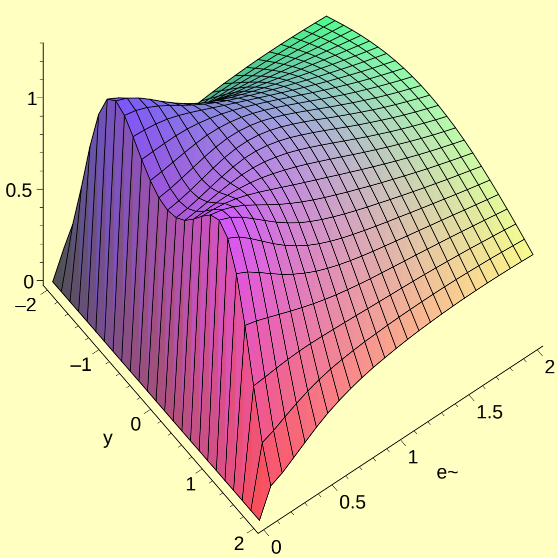



w(x,1,1,e)
4e+10
2e+10
0
–2
–1
x
0
1
2
0
0.5
1
1.5
2
e~
<exp(-s(y,1,1,e))>


w(x,–1,1,e)
40000
20000
0
–2
–1
x
0
1
2
0
0.5
1
1.5
2
e~
<exp(-s(y,–1,1,e))>


1
0.5
0
–2
–1
y
0
1
2
0
0.5
1
1.5
2
e~


1
0.5
0
–2
–1
y
0
1
2
0
0.5
1
1.5
2
e~

**Conclusion:**

This process can be interpreted as a way to implement block variables straight into the integrand, by means of smoothing it out. One big open question, under this view (coarse-graining, the graining parameter being $\epsilon$), regards the renormalization procedure.

*We have just begun to calculate!*