Alternative Numerical Techniques

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"Traditional" Monte Carlo approaches have produced many good results but they do have limitations.

- Fermions are hard;
- Non-positive definite actions are hard;
- Rapid oscillations, phase transitions are hard;
- Symmetry breaking is hard; &
- Scattering is hard.

Good answers require immense resources.

For these reasons it is worth examining alternative approaches.
We have been working on two:

- Nested approximations to the Schwinger-Dyson equations;
  - Source Galerkin;
- A tuned Monte Carlo method, SPMC (Stationary Phase Monte Carlo) — our SPMC ideas originated with Jimmie Doll and chemistry collaborators.

Source Galerkin is particularly powerful because it can be defined on the continuum. The idea is very simple:

- Assume that the QFT action is written with sources $J_i(x)$ for every field $\phi_i(x)$,

- The amplitude $\mathcal{Z} = \langle +0|0- \rangle_{J_i}$ satisfies the differential equation

$$\underbrace{F\left(\frac{\delta}{\delta J_i(x)}\right)}_{\substack{\text{essentially} \\ \text{the field} \\ \text{equations}}} \mathcal{Z} = 0$$

A familiar example is the ϕ^4 scalar QFT,

$$(\partial^2 + m^2)\,\phi(x) + g\,\phi^3(x) = J(x)\ ,$$

which becomes (Euclidian space)

$$\left[(\partial_x^2 + m^2)\,\frac{\delta}{\delta J(x)} + g\left(\frac{\delta}{\delta J(x)}\right)^3 - J(x)\right]\mathcal{Z}[J] = 0\ .$$

This is a very non-trivial infinite set of coupled differential equations. To get some idea of its complexity, look at the one point (0-dim) case,

$$\left[m^2\,\frac{d}{dJ} + g\left(\frac{d}{dJ}\right)^3 - J\right]\mathcal{Z}[J] = 0\ .$$

Even this simple equation has 3 independent solutions and requires the input of 3 parameters for a full solution.

The 3 solutions can be characterized by their degree of singularity:

- Regular at $g \to 0$: consistent with perturbation theory;
- Singular ($\sim g^{-1/2}$) at $g \to 0$: symmetry breaking; &
- Singular ($\sim \exp(m^2/4\,g)$) at $g \to 0$: instanton.

In the path integral language they correspond to the 3 saddle points of the path integral in the complex ϕ-plane.

Any numerical computation must be done with great care.

The Schwinger condition,

$$\frac{d\,\mathcal{Z}}{d\,g} = \int (dx) \left(\frac{\delta}{\delta J(x)} \right)^4 \mathcal{Z}\,,$$

can be used to stabilize the phase choice in iterative approximations.

In general, *How do you, numerically, estimate these equations?*
A very direct approach is to just assume a functional expansion in the sources and introduce some way to minimize the error.
Of course, the correct answer is given as,

$$\mathcal{Z} = \exp\left\{ A_0 + \int A_i(x)\, J_i(x) + \int A_{ij}(x,y)\, J_i(x)\, J_j(y) + \cdots \right\}$$

The problem is to simplify the calculations of the A_i's

In order to do so, we put information we already have about the field theory: spectral information, relativity, Kallen Lehman, representations, . . .

Make decompositions by assuming that we can represent $\mathcal{Z}$ in terms of two point functions G and all possible higher order graph like terms constructed from products made from $G(x-y)$ but with possibly different parametrizations. For example,

$$\begin{matrix}\text{4th order}\\ \text{term is}\end{matrix} = J(x_1)\, J(x_2)\, G^{(4)}(x_1, x_2, x_3, x_4)\, J(x_3)\, J(x_4)$$

Even without equations of motion $G^{(4)}$ is not arbitrary, it must be consistent with all symmetries present for $J_i(x) = 0$: Lorentz invariant, space-time translation.

Span this space (or a very large part of it) with free field propagators.

$$J\,J\,G_4\,J\,J =$$

where we include all possible graphs with 4 external lines.
Here the propagator is,

$$G(x-y) = \int dK^2\,dk\,\frac{e^{i\,k(x-y)}}{-k^2+K^2}\,a_p(K^2)$$

This satisfies the usual space-time restrictions. $a_p(K^2)$ may be different for every propagator.

The above expansion inserted into the source field equations yield conditions of constraint on all the $a_p(K^2)$.

In general, this is too complicated to directly solve, i.e., we cannot exactly solve field theory.

There are cross-terms in any equation and the equations are non-linear!

Simplify!

- Truncate the expansion in J;
- Limit the number of masses in each propagator;
- Limit the number of graphs considered for each J.

Do this in an organized, systematic, way so that after a first guess more terms can be included so as to iterate answers.

Petrov will show some more detail about this, but roughly it works as follows: Any approximate $\mathcal{Z}$ satisfies,

$$F\left(\frac{\delta}{\delta J}\right) \mathcal{Z}_{\text{approx}}[J] \neq 0 \; .$$

The idea of Source Galerkin is to require that,

$$\int dJ_1 \cdots dJ_n \, H_i(J_1, \ldots, J_n) \, F\left(\frac{\delta}{\delta J}\right) \mathcal{Z}_{\text{approx}}[J] = 0 \; ,$$

so that $\mathcal{Z}_{\text{approx}}[J]$ satisfies the field equations "on the average".

The number of H_i is picked so that all the undetermined weight in $\mathcal{Z}_{\text{approx}}[J]$ can be calculated.

In general, the equations to be solved are nonlinear. This needs to be done in a very careful and systematic manner, there are parameters to be tuned!

Theorems for simpler Galerkin approaches promise convergence and we assume this is true for QFT.

We have done this in many models (with amazing accuracy when a check is available).

However, the results must be confirmed to be stable and convergent.

Higher iterations, while in principle not hard, has caused us computational difficulties.

For this reason we have done 2 things:

- Re-examined perturbation theory, using variants of our numerical techniques,

- Re-analyzed trivial 0-dimensional models for higher order Galerkin expansion, Petrov will talk a bit more about this.

Perturbation Theory:

The usual graphical rules apply (not our extended rules).

New way to numerically calculate graphs.

Sinc-function expansion:

$$S_k(h,x) \equiv \frac{\sin(\pi\,(x-k\,h)/h)}{\pi\,(x-k\,h)/h}\,, \qquad k \in \mathbb{Z}\,;$$

cutoff propagator

$$G_\Lambda(x) = \int \frac{d^4p}{(2\,\pi)^4}\,\frac{e^{i\,p\,x}}{p^2+m^2}\,e^{-p^2/\Lambda^2}\,.$$

In terms of the sinc expansion this becomes,

$$G_{\Lambda,h}(x) = \frac{m^2\,h}{(4\,\pi)^2}\sum_{k=-\infty}^{\infty} p(k)\,\exp\left[-\frac{m^2\,x^2}{4\,C(k)}\right]$$

$$C(k) = e^{k\,h} + \frac{m^2}{\Lambda^2}$$

$$p(k) = \frac{e^{k\,h}-e^{k\,h}}{C^2(k)}\,.$$

Typically, the propagator can be approximated to very high accuracy (1 part in 10^{16}).

With fewer than 100 terms in the sum, similar statements hold for fermions.

This is the best way to numerically calculate perturbation theory. It is incredibly quick and accurate.

We have calculated to 4th non-trivial order (3 loops) and successfully responded to challenges to the accuracy and speed of the method.

This is the way to check the muon magnetic moment calculation.

Technical issues with accuracy using an "auto" renormalization scheme have slowed us down.

The ease and speed of these calculations is why we believe we can iterate source Galerkin to moderate order with relatively small amounts of computer time.