Dimensionality: A Powerful Tool for Understanding and Calculating in Science

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ne first example of a calculation that requires you to change the number of dimensions in ur problem in order to get to the answer comes around with the integral:

$$
\begin{aligned}
\mathcal{I} &= \int_{-\infty}^{\infty} \mathrm{exp}\{-x^2\}\, dx \ ; \\
\therefore\ \mathcal{I}^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{exp}\{-x^2 - y^2\}\, dx\, dy \ ; \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} \mathrm{exp}\{-r^2\}\, r\, dr\, d\theta \ ; \\
&= \pi \ . \\
\Rightarrow\ \mathcal{I} &= \sqrt{\pi} \ .
\end{aligned}
$$

nat is, in order to solve a **1**-dimensional problem we had to resort to going to a **2**-dimensional e!

milar ideas apply to many areas of science, particularly physics! ☺

- General Relativity (Kaluza-Klein);
- Quantum Field Theory (dimensional regularization, dimensional [de]construction); &
- String Theory (extra dimensions).

R: Kaluza-Klein

aluza-Klein theory was very popular in the 1970s, in part because the most general supergrav- y theory is most easily written in eleven dimensions. It can be generalized beyond the original rmulation to include not only electromagnetism, but other gauge theories as well. (Bryce eWitt showed this in 1963, as an exercise [!] in his Les Houches lectures.) These days it's rly widely regarded, in a somewhat modified form, as an **element** of other theories. String eory, for example, requires more than four spacetime dimensions, and the "extra" dimensions e usually dealt with by "compactification" a la Kaluza-Klein. On the other hand, it's generally jected as the ultimate answer to unification, though a few people are still fond of it. There e two main problems:

1. It's basically general relativity, and to make sense of it as a quantum theory, you would have to understand how to quantize general relativity. No one knows how to do that.

2. Witten showed twenty years ago that standard Kaluza-Klein theories are incompatible with the observed existence of chiral (left- or right-handed) fermions. There have been suggestions of ways out, for example by adding torsion or by looking at fermions as nonelementary solitons, but none has looked very appealing.

R: Dimensionally Reduced Gravitational Chern-Simons Term and its Kink

hen the gravitational Chern-Simons term is reduced from 3 to 2 dimensions, the lower mensional theory supports a symmetry breaking solution and an associated kink. Kinks in neral relativity bear a close relation to flat space kinks, governed by identical potentials.

should also be noted that Chern-Simons theory is **intimately** related to what is known **Donaldson Invariants** and, as such, to the more recent **Seiberg-Witten Theory**. rthermore, the **Chern character** is related to the **[Atyiah-Singer] index theorems**!

Gravitational Instantons: complete d-dimensional Riemannian manifolds, (M, g), nich are either *Einstein* or *vacuum* satifying one of the following boundary conditions:

1. M is compact without boundary;
2. (M, g) is `asymptotically locally Euclidean`, ALE, i.e., it has an "infinity" which is like that of E^d; &
3. (M, g) is `asymptotically locally Flat`, ALF, i.e., it has an "infinity" which is like that of E^{d-1} with the last direction being periodic.

Yang-Mills Instantons: Given an action of the type,

$$S^{\mathsf{E}}_{\mathsf{YM}}[\mathcal{A}] = -\frac{1}{2}\int_M \mathrm{tr}(\mathcal{F}\wedge *\mathcal{F}) \ ;$$

$$\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = \frac{1}{2}\mathcal{F}_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \ ;$$

here $\mathcal{A}$ is the `connection` 1-form with values in a given gauge group, $\mathcal{F}$ is the `curvature` form and the Hodge-$*$ operator is taken with respect to the Euclidean metric (i.e., $** = 1$). evaluate this action it's important to find the *local minima* of the [Euclidean] action and mpute the quantum fluctuations around them. An *instanton* is either a self-dual or an anti-lf-dual field strength: $\mathcal{F} = \pm * \mathcal{F}$. Furthermore, the "big deal" about such a thing is the ct that it does **not** depend on the Hodge-$*$ anymore, i.e., the action is independent of the gnature of the] metric! This means that this is a **topological beast**. ☺

Gravitational Chern-Simons: One should start by noting that: $\mathrm{d}\,\mathrm{tr}(\mathcal{F}^2) = 0$. There-re, by Poincaré's Lemma, $\mathrm{tr}(\mathcal{F}^2)$ is **locally exact**: $\mathrm{tr}(\mathcal{F}^2) = \mathrm{d}\,K$. Thus, the closed rm, $\mathrm{tr}(\mathcal{F}^2)$, is identified with the **second Chern character** and K to the **Chern-Simons rm**: $K = \mathrm{tr}(\mathcal{A}\ \mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A})$. The link with *gravity* comes when one identifies the ristoffel connection with $\mathcal{A}$ (the "gauge" connection), in the following way: $\Gamma^{\lambda}_{\mu\nu} = [\mathcal{A}_\mu]^{\lambda}_{\nu}$! at means that ν and λ are *internal indices* (gauge group), while μ is a spacetime index.

Dimensional Reduction: To effect the dimensional reduction from d, $(t, x_1, \ldots, x_{d-1})$, $d-1$, $(t, x_1, \ldots, x_{d-2})$, dimensions, we choose the Lorentz signature and set the Kaluza-ein *ansätz* for the metric tensor G:

$$G_{\mu\nu} = \phi \begin{pmatrix} g_{\alpha\beta} - a_\alpha a_\beta & -a_\alpha \\ -a_\beta & -1 \end{pmatrix} ,$$

here $g_{\alpha\beta}$ is the metric tensor for the $d-1$ spacetime, a_α is a $d-1$ vector and ϕ is a alar. Furthermore, because the Chern-Simons term is `conformally invariant`, it does **ot** depend on ϕ, which we henceforth set to 1.

hus, going from 3 to 2 dimensions, it's not hard to find that the gravitational Chern-Simons rm is given by,

$$\text{CS} = -\frac{1}{8\pi^2} \int \sqrt{-g}\,(f\,r + f^3) \; ;$$
$$g = \det(g_{\alpha\beta}) \; ;$$
$$f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha = \sqrt{-g}\,\epsilon_{\alpha\beta}\,f \; ;$$

$\epsilon_{\alpha\beta}$: totally antisymmetric 2-dim symbol with $\epsilon_{0\,1} = 1$;

r : 2-dimensional curvature.

Chern-Simons Dynamics: If the entire dynamics is governed by the Chern-Simons term, e get two types of solution:

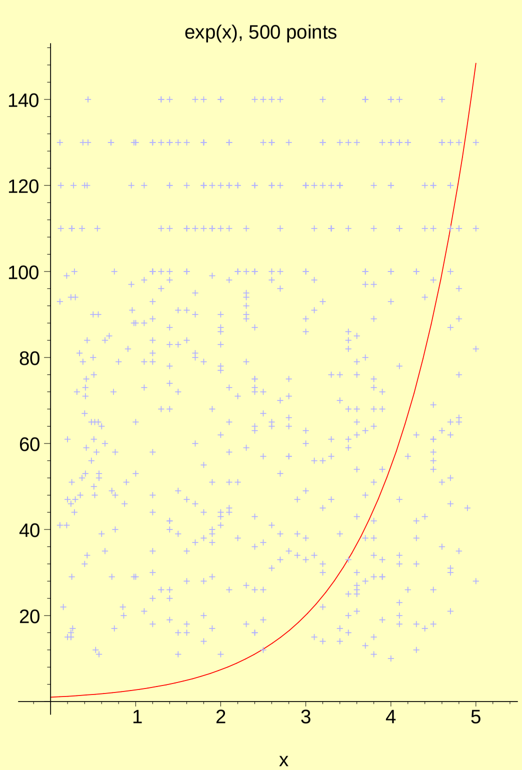
1. **symmetry preserving**; &
2. **symmetry breaking**!

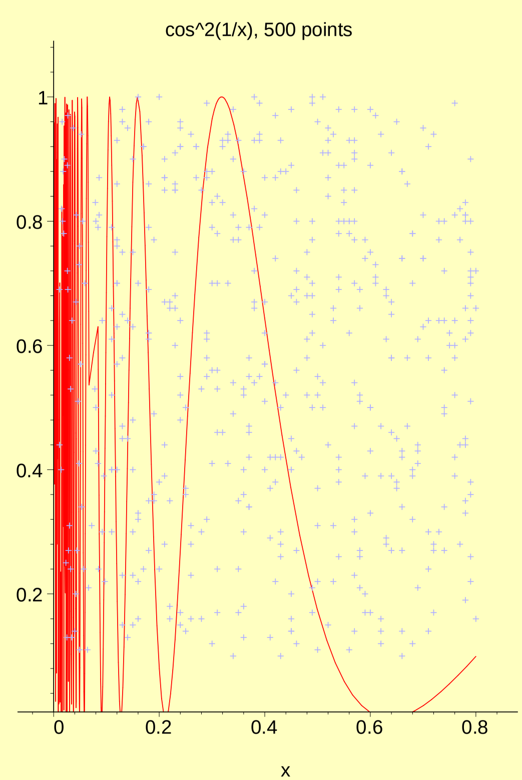
though both show constant curvature, the latter is `maximally symmetric`.

hen the symmetry breaking solution is present, there's also a **kink** solution, which interpo-tes between the two symmetry breaking solutions!

FT: Mollification in 0-dimensions

his method tries to address the well known question of Monte Carlo simulations of highly cillatory functions, as illustrated below:





- **Convolution:** It's a mathematical operation defined as,

$$f,\, g \in \mathscr{S}(\mathbb{R}^n) \;\Rightarrow\; (f * g)(y) = \int_{\mathbb{R}^n} f(y-x)\, g(x)\, dx\ .$$

hus, it is not difficult to see that the convolution is somewhat like taking an average of f th respect to the weighting function g. Among the important properties of convolutions at we'll be using are:

$$f * g = g * f\ ;$$

$$(f * g) * h = f * (g * h)\ ;\ \text{if } \textbf{all} \text{ except one of } f, g, h \text{ have } \textbf{compact support}.$$

$$\therefore\ \int_{\mathbb{R}^n} (f * g) = \int_{\mathbb{R}^n} f\ \int_{\mathbb{R}^n} g\ ;\ \textbf{iff } u, v \text{ have compact support.}$$

- **Approximate Identity (aka Mollifiers):** It is a **positive** C^∞ function, (denoted by x)), whose support lies in the sphere of unity radius about the origin in $\mathbb{R}^n$, satisfying $j(x)\, dx = 1$. The sequence of functions $j_\epsilon(x) = \epsilon^{-n}\, j(x/\epsilon)$ is called an **approximate entity**.

Mollification: This technique consists of using convolutions in order to smooth *"badly ehaved functions"*. (On a more mathematical note, the mollifier technique is commonly used order to generate functions from distributions/generalized fuinctions.) Basically, the method nsists of using a convolution of an **approximate identity** with the [generalized] function at we'd like to smooth. Thus, if $f : U \longrightarrow \mathbb{R}$ is **locally integrable**, the its mollification is fined to be:

$$f_\epsilon = j_\epsilon * f \ ;$$

$$f_\epsilon(x) = \int_U j_\epsilon(x-y)\, f(y)\, dy = \int_{B(0,\epsilon)} f(x-y)\, j_\epsilon(y)\, dy \ .$$

Properties of Mollifiers:

1. $f_\epsilon \in C^\infty$;
2. $\lim_{\epsilon \to 0}(T * j_\epsilon) = T$, **weakly** (almost everywhere), if T is a continuous and linear Schwartz function; &
3. If $f \in C^\infty(U)$, then $\lim_{\epsilon \to 0} f_\epsilon = f$ **uniformly** on **compact** subsets of U.

0-dim QFT: To illustrate some features of QFT, let's consider spacetime to be 0-dim. everal features of higher dimensional QFTs can already be observed with this somewhat mple minded models. We have been analyzing those in order to generalize its results later.

$$L[\phi] = -\frac{1}{2}\phi^2 - \frac{g}{4!}\phi^4\,;$$

$$\mathcal{Z}[g,j] = \int \exp\left\{-\frac{1}{2}\phi^2 - \frac{g}{4!}\phi^4 + j\,\phi\right\} d\phi\,.$$

hus, by making the change of variables $\phi \mapsto \phi/g^{-1/4}$, it is not difficult to see that, ought as a function of g, $\mathcal{Z}[g,j]$ *converges* to an **analytic** function for $g \neq 0$. Also, there an *essential singularity* and a 4th-order branch point at $g = 0$. At this point, let's pretend at one did not know about this and one attempted to expand this function as a power series g:

$$\exp\left\{-\frac{g}{4!}\phi^4\right\} = \sum_{m=0}^{\infty}\frac{(-g)^m\,\phi^{4m}}{(4!)^m\,m!}\,;\quad \exp\{j\,\phi\} = \sum_{2k=0}^{\infty}\frac{j^{2k}\,\phi^{2k}}{(2k)!}\,.$$

$$\mathcal{Z}[g,j] = \int \sum_{m,2k=0}^{\infty}\frac{(-g)^m\,j^{2k}}{(4!)^m\,m!\,(2k)!}\,\phi^{4m+2k}\,\exp\left\{-\frac{1}{2}\phi^2\right\} d\phi\,. \qquad (1)$$

0-dim QFT (cont'd): Note that,

$$\int \phi^{2n} \exp\left\{-\frac{1}{2}\phi^2\right\} d\phi = \frac{(2n)!}{2^n\, n!} \underbrace{\int \exp\left\{-\frac{1}{2}\phi^2\right\} d\phi}_{=\sqrt{\pi/2}} .$$

n order to give a graphical interpretation to the numbers that show up above, let's re-write
) (and note that, the integral can commute with the sum given that everything *converges*
this particular case) in the following form:

$$\mathcal{Z}[g,j] = \sum_{m,2k} \int \frac{\overbrace{\frac{(-g)\,\phi^4}{4!}\cdots\frac{(-g)\,\phi^4}{4!}}^{m \text{ terms}}}{m!} \cdot \frac{\overbrace{j\,\phi\cdots j\,\phi}^{2k \text{ terms}}}{(2k)!} \cdot \exp\left\{-\frac{1}{2}\phi^2\right\} d\phi .$$

aïvely, one could write a graph with $4\,m + 2\,k$ vertices and join them in pairs, what would
ve the correct answer. On the other hand, one could notice that there are two different
ypes" of ϕ: the ones that come from the ϕ^4 term and the ones that come from the $j\,\phi$
ource) term. To keep track of this in a graphical way, let's introduce two types of vertices:

1. m vertices with valence 4, which correspond to the $m\,\phi^4$ terms; &
2. $2\,k$ vertices with valence 1, which correspond to the $2k\,j\phi$ terms.

0-dim QFT (cont'd): Thus, the integral can be replaced by a sum over graphs, since e keep track of the g's, j's and combinatorial factors. Now, consider the action of the group hich permutes vertices and edges: this group has order $(4!)^m\, m!\,(2\,k)!$. This is exactly e combinatorial factor that should be attached to each graph. Thus, the Orbit-Stabiliser eorem can be used and the sum over all graphs can be replaced by the *sum over isomorphism asses* of graphs *weighted* by $1/\,\mathsf{Aut}(G)$. In this manner, one has the following graphical aluation of the integral:

- $\forall$ *isomorphism class* of graphs, associate:
  - $(-g)$ for each vertex of valence 4;
  - (j) for each vertex of valence 1; &
  - $[1/\,\mathsf{Aut}(G)]$.

this point one remembers that the expansion performed was around an *essencial singularity* d, so, is **not** valid as a power series. Thus, it will be shown now that it is an asymptotic pansion for the integral. For the sake of simplicity, let's do this for $\mathcal{Z}[g,0]$, i.e., ignore the urce terms. In order to do so, just recall that,

0-dim QFT (cont'd):

$$\left|\exp(x) - \sum_{n=0}^{k} \frac{x^n}{n!}\right| \leqslant \frac{x^{k+1}}{(k+1)!}\,.$$

his means that the error in computing the integral for $\mathcal{Z}[g,0]$ by taking the first k terms the series obtained above is bounded by $C\,g^{k+1}$. C is a constant that depends on k but t on g. This shows that the series is an asymptotic expansion for the integral*

Borel Summation: At this point, the series that was computed for $\mathcal{Z}[g, j = 0]$ is of the rm: $\sum_n a_n\, x^n$, with $a_n \approx C\, n!$ and there are known ways to sum series of this form, one of hich is **Borel Summation**. The trick is the following: $x^{n+1}\, n! = \int_0^\infty t^n \exp\{-t/x\}\, dt$ nce $\sum_n a_n\, x^n = \int_0^\infty \exp\{-t/x\} \left(\sum_n \frac{a_n}{n!} t^n\right) \frac{dt}{x}$. However, in QFT, there are a couple problems with this method: (1) It does **not** pick up *non-perturbative* effects; & (2) $g(t) \equiv$ $_n \frac{a_n}{n!}\, t^n$ may have singularities on $t > 0$.

n asymptotic expansion is something that becomes more accurate as $g \longrightarrow 0$ with k fixed. A power series s something which becomes more accurate as $k \longrightarrow 0$ with g fixed.

→ **Mollifiers and 0-dim QFT:** The idea is to mollify the *Partition Function* in order to be
le to perform Monte Carlo computations in Minkowski spacetime! ☺

$$\mathcal{Z}[j] = \frac{\int\limits_U \left\{ \int\limits_\Gamma j_\epsilon(\varphi - \phi)\, e^{i\,S[\phi] + i\,j\,\phi}\,[d\phi] \right\} [d\varphi]}{\int\limits_U \left\{ \int\limits_\Gamma j_\epsilon(\varphi - \phi)\, e^{i\,S[\phi]}\,[d\phi] \right\} [d\varphi]}\,,$$

ote that U is the domain of the mollifier, while Γ is the domain of the action. It is important realize that Γ need **not** be the real line! Different domains of integration will give rise to fferent solutions of the field equations, therefore enabling us to pick distinct **phases** of the eory. (Note that the choice of domain of integration is intimately related to the definition the **measure** of the path integral. Furthermore, it's completely analogous to choosing undary conditions for the Schwinger-Dyson equations.) This is of utmost importance for **mmetry breaking**! ☺

[Monte Carlo] Simulations: In order to better perform the MC simulation we'll perform importance sampling and it's very convenient to choose a *sampling/importance function*, W_ϵ, that better resembles the integrand that we're dealing with. The job of the importance mpling is to pick out the most contributing pieces of the integrand and sample them better an the pieces that do not contribute all that much for our integration.

hoosing a Gaussian mollifier (see below), our choice of sampling function is (note that it hounts to performing a saddle-point/stationary phase expansion in the above integrand):

$$j_\epsilon(x) = \frac{e^{-(x/\epsilon)^2/2}}{\sqrt{2\,\pi}\;\epsilon}\;;$$

$$W_\epsilon(\varphi) = \left|\frac{\exp\Big\{i\,S(\varphi_0) - \frac{1}{2}\,S'\cdot(1+\epsilon\cdot S''\cdot\epsilon)^{-1}\cdot S'\Big\}}{\sqrt{1+\epsilon\cdot S''\cdot\epsilon}}\right|\;.$$

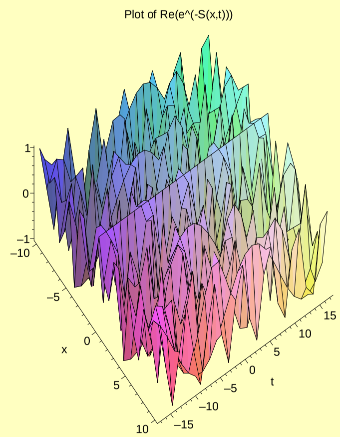
Examples: In what follows, we'll show how this method works with the ϕ^3- and ϕ^4-eories (in 0-dim).

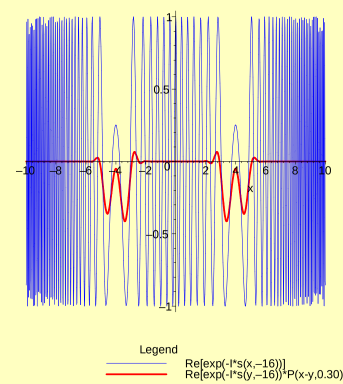
Airy Function:

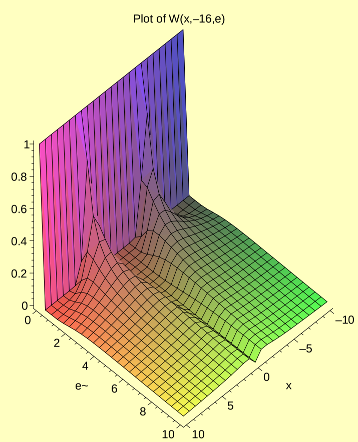
$$\mathcal{Z}[J] = \frac{\int\limits_{-\infty}^{\infty} \exp\left\{ i\,\frac{\phi^3}{3} + i\,J\,\phi \right\} d\phi}{\int\limits_{-\infty}^{\infty} \exp\left\{ i\,\frac{\phi^3}{3} \right\} d\phi} \equiv \frac{\mathrm{Ai}(J)}{\mathrm{Ai}(0)}\,, \tag{2}$$

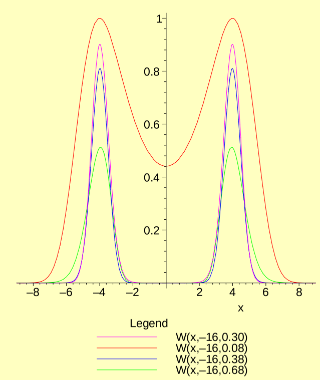
$$\mathcal{Z}[J] = \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} j_\epsilon(\varphi - \phi)\, \exp\left\{ i\,\frac{\phi^3}{3} + i\,J\,\phi \right\} d\phi\, d\varphi}{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} j_\epsilon(\varphi - \phi)\, \exp\left\{ i\,\frac{\phi^3}{3} \right\} d\phi\, d\varphi} \equiv \frac{\mathrm{Ai}(J)}{\mathrm{Ai}(0)}\,. \tag{3}$$

Airy Function (cont'd):







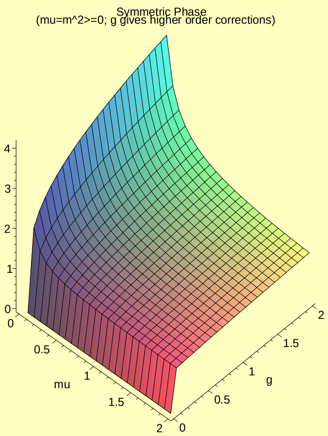


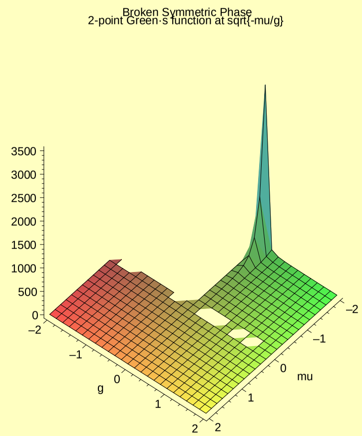
ϕ^4-theory:

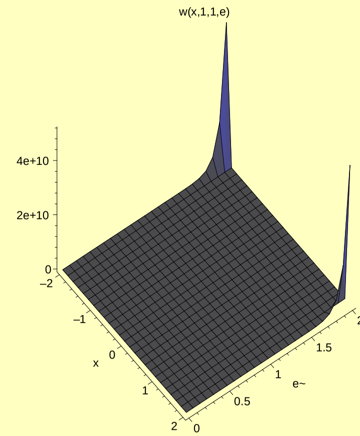
$$\mathcal{Z}[J] = \frac{\int_{-\infty}^{\infty} \int_{\Gamma} j_\epsilon(\varphi - \phi)\, \exp\left\{ i\,\frac{\mu}{2}\,\phi^2 + i\,\frac{g}{4}\,\phi^4 - i\,J\,\phi \right\} d\phi\, d\varphi}{\int_{-\infty}^{\infty} \int_{\Gamma} j_\epsilon(\varphi - \phi)\, \exp\left\{ i\,\frac{\mu}{2}\,\phi^2 + i\,\frac{g}{4}\,\phi^4 \right\} d\phi\, d\varphi} \,.$$

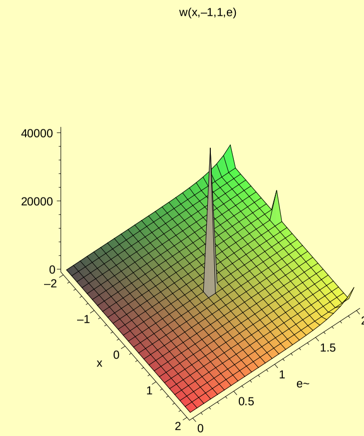
s usual, its parameters, (m, g), will determine the phase structure of the theory. The llowing are the plots of the 2-point Green's function in the symmetric $(\mu \geqslant 0)$ and broken-mmetric $(\mu < 0)$ phases, respectively.

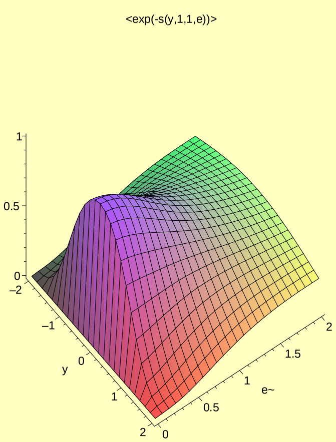
ϕ^4-theory (cont'd):

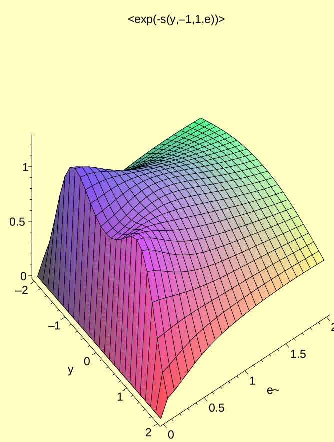












FT: Source-Galerkin in 0-dimensions

The Galerkin Technique: Given a differential equation $D[u] = 0$ an approximate lution can be obtained via the following method:

$$u_a = \sum_{i=0}^{N} a_i \phi_i(x) \; ;$$

Residue Equation: $R(a, x) = D(u_a) \neq 0$.

olving the equation is equivalent to determining such set of coefficients $\{a\}$ that,

$$R(a, x) = 0 \; .$$

The Galerkin Technique (cont'd):

- Define inner product of two functions;
- Choose a set of weight functions w_i;
- A set of equations is constructed by requiring that inner product of the residue with weight functions is equal to zero: $(R, w_i) = 0, \quad i = 0, \ldots, N$;
- Provided that w_i are members of a complete set of functions the procedure outlined above guarantees that as $N \to \infty$ approximate solution u_a converges to the exact solution u in the mean; &
- The procedure outlined above is called the method of weighted residues. A specific choice of weight functions $w_i = \phi_i$ is called Galerkin method.

The Galerkin Technique (cont'd):

- We know from mathematics that Galerkin technique produces approximate solution, which converges to the exact solution as number of terms goes to infinity;

- In practice, the series has to be truncated. It likely that for nontrivial theories calculations for $N > 8$ will be impossible to perform. So we must ensure rapid convergence of the method and that high accuracy can be achieved even with a few terms in the ansatz; &

- The method in question produces approximate solution which converges to the exact solution in the mean. This implies that the accuracy of the result depends on the choice of inner product.

The Ultralocal ϕ^4:

$$\mathcal{L} = \frac{g}{4}\phi^4 + \frac{\mu}{2}\phi^2 - j\phi \; ;$$

$$0 = \left(\mu\frac{d}{dj} + g\frac{d^3}{dj^3} - j\right) Z \; ;$$

$$Z(j) = Z(0)\left(\sum_{k=0}^{\infty} \frac{1}{k!} G_k\, j^k\right) \; ;$$

$$0 = g\, G_{n+3} + \mu\, G_{n+1} - n\, G_{n-1} \; ;$$

$$G_{2n} = (2n-1)!!\, \frac{U(n,t) + (-1)^n \rho\, U(n,-t)}{U(0,t) + \rho\, U(0,-t)} \; ;$$

$$G_{2n+1} = \frac{2n!!}{n!}\,(-t)^n\, \frac{V(n+\frac{1}{2},t)}{V(\frac{1}{2},t)}\,\alpha\, \frac{t^{\frac{1}{2}}\, e^{\frac{t^2}{4}}}{U(0,t) + \rho U(0,-t)} \; ;$$

where $t = \dfrac{1}{\sqrt{2g}}\; ;\; \mu = 1$

The Numerical Solution: In analogy with the theoretical solution, a truncated power ries can be chosen as an ansatz.

$$Z_a(j) = \sum_{k=0}^{N} \frac{1}{k!} a_k j^k$$

Residual equations are obtained using one of the inner products presented below.

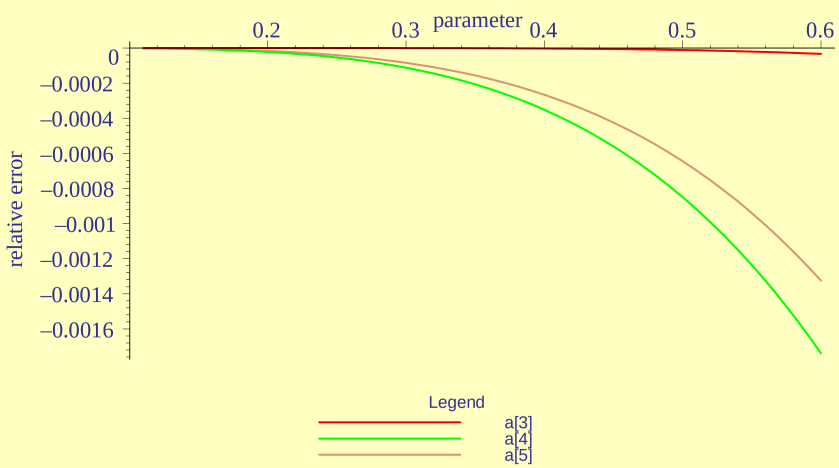
1.

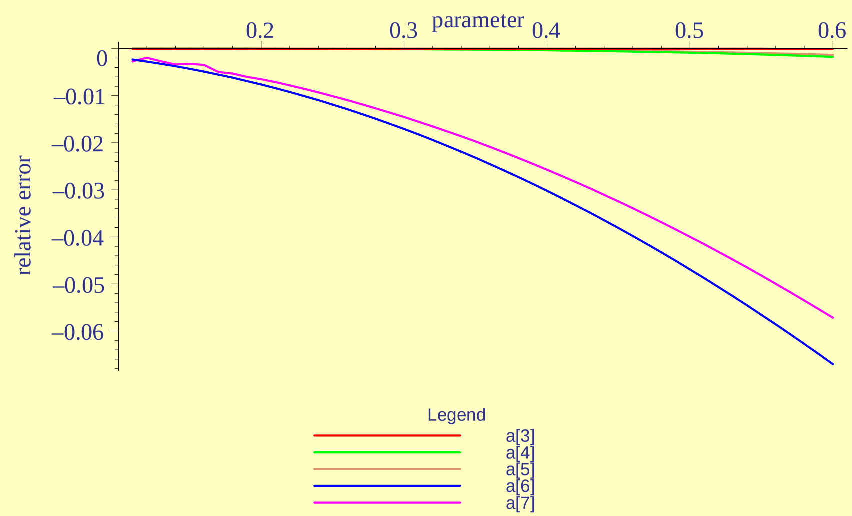
$$\int_{-c}^{c} j^n R(j)\,dj, \quad n = 0, \ldots, N-3.$$

2.

$$\int_{-\infty}^{\infty} j^n R(j) \exp\left[-\frac{j^2}{\epsilon^2}\right] dj, \quad n = 0, \ldots, N-3.$$

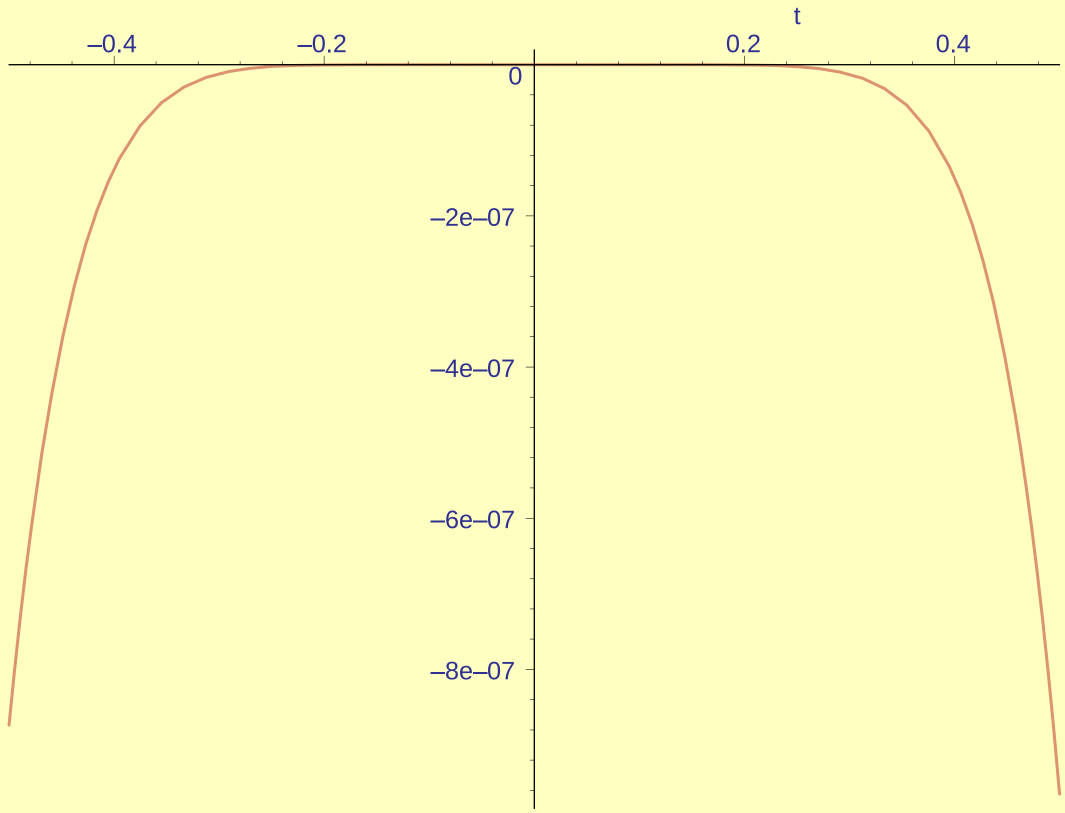
The Numerical Solution (cont'd):





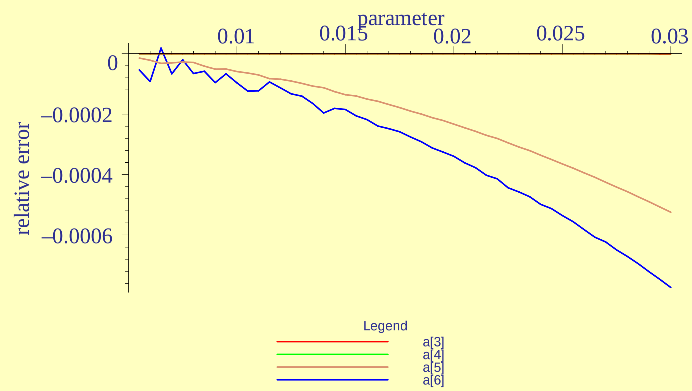
der by order relative error is plotted for N=7 and residual equations derived from $\int_{-c}^{c} j^n R(j)\,dj$ with respect to the range of integration c.

The Numerical Solution (cont'd):

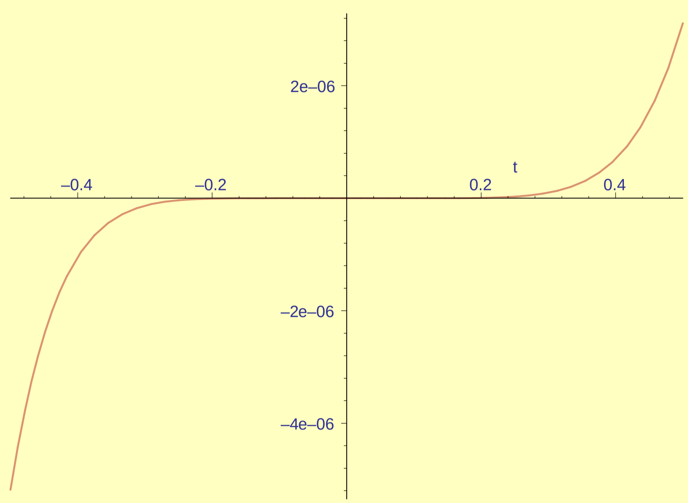


Absolute error in the generating functional $Z_a(j)$ is plotted with respect to the value of j.

The Numerical Solution (cont'd):



Order by order relative error is plotted for $N = 6$ with respect to the parameter ϵ.



Absolute error in the generating functional $Z_a(j)$ is plotted with respect to the value of j.Residual equations derived from

$$\int_{-\infty}^{\infty} j^n R(j) \exp\left[-\frac{j^2}{\epsilon^2}\right] dj \ .$$

The Numerical Solution in Hermite Polynomials:

set of Hermite polynomials is defined by

$$H_n(\xi) = (-1)^n \exp\left\{\frac{\xi^2}{\epsilon^2}\right\} \frac{d^n \exp\left\{-\frac{\xi^2}{\epsilon^2}\right\}}{d\xi^n}$$

Or by a recursion relation

$$\begin{aligned} H_0(\xi) &= 1, \\ H_{n+1}(\xi) &= \frac{2\xi}{\epsilon^2} H_n(\xi) - H_n'(\xi). \end{aligned}$$

These polynomials are orthogonal under following inner product

$$\int_{-\infty}^{\infty} H_k(x) H_l(x) \exp\left(-\frac{x^2}{\epsilon^2}\right) dx = \delta^{kl}$$

The Numerical Solution in Hermite Polynomials (cont'd):

ow the trial function for the generating functional can be written as

$$Z_a(j) = \sum_{i=0}^{N} a_i H_i(j).$$

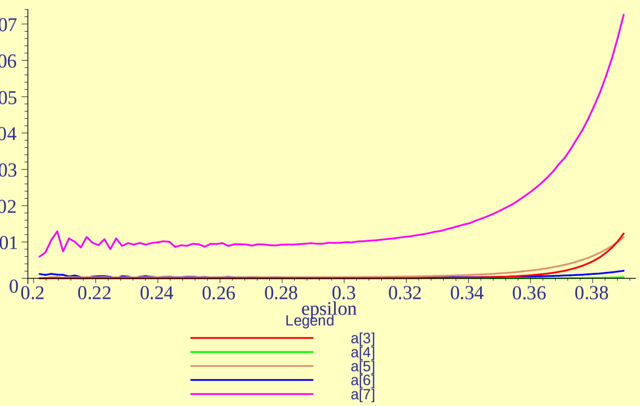
sidue is computed by substituting the above expression in to Schwinger Dyson equations.

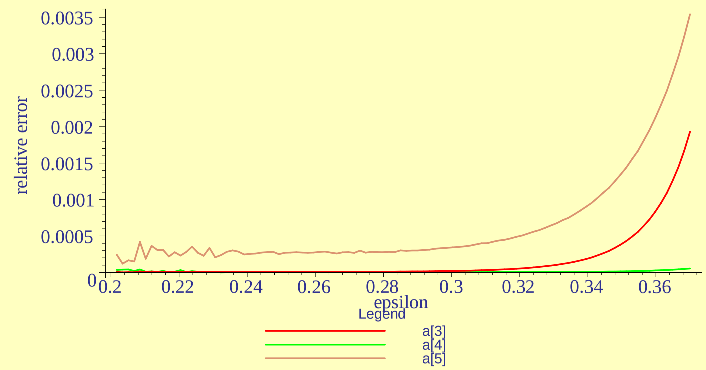
$$R(j) = \sum_{i=0}^{N} a_i \left(\mu \frac{d}{dj} H_i(j) - g \frac{d^3}{dj^3} H_i(j) + j H_i(j) \right).$$

nally, a set of equations is obtained by requiring that the residue is orthogonal to a set of hermite polyno-
als.

$$\int_{-\infty}^{\infty} \sum_{i=0}^{N} a_i \left(\mu \frac{d}{dx} H_i(x) - g \frac{d^3}{dx^3} H_i(x) + j H_i(x) \right) \times H_k(x) \exp\left(-\frac{x^2}{\epsilon^2} \right) dx = 0.$$

The Numerical Solution in Hermite Polynomials (cont'd):





ler by order relative error is plotted for $N = 7$ and $N = 5$. Residual equations derived from $\int_{-\infty}^{\infty} H_n(j) R(j) \exp\left[-\frac{j^2}{\epsilon^2}\right] dj$ with respect to the parameter ϵ.