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Computational High Energy Physics

Daniel Doro Ferrante and Gerald S. Guralnik

http://chep.het.brown.edu/

Brown University, BOX 1843

{danieldf,gerry}@het.brown.edu

1. Source-Galerkin

1.1 The Galerkin Technique

The Source-Galerkin technique consists of the application of the Gelerkin technique to try to solve the Schwinger-Dyson equations. The Galerkin technique can be easily understood from the following example: Let $L[u]=0$, where L is a linear operator and u is unknown. We try to solve for u selecting a set of basis functions, $(u_1,\dots,u_n,\dots)$, such that $u\approx\sum_i c_i\,u_i$. Since this set of functions is arbitrarily chosen, what we want is to minimize the following expression: $c_i\,L[\sum_i u_i]=\mathrm{Res}(u)$, *i.e.*, we want to find the set of u_i's that minimizes the residues $\mathrm{Res}(u_i)$. In principle, if we are able to find u_i's such that $\mathrm{Res}(u_i)=0$, then $u\equiv\sum_i c_i\,u_i$.

This method has the advantage of not having to deal with fermionic integrals, treating both, bosons and fermions, on a symmetrical footing. Furthermore, phases are easily handled.

1.2 Applications to Quantum Field Theory

In the context of QFT's, this takes the following form: once all the information of the theory is contained on its Generating Functional, $\mathcal{Z}[J]$, we want to use the above technique to solve Schwinger-Dyson's equation,

$$\frac{\delta S[-i\,\delta/\delta J]}{\delta\phi}\mathcal{Z}[J]+J(x)\,\mathcal{Z}[J]=0\;.$$

(That is, this is equivalent to substituting $\phi\mapsto -i\,\delta/\delta J$ in the action and to expanding the generating functional, requiring the equations of motion to be the solutions that extremizes the action.)

Our ansätze will be based on our knowledge of the Green's functions. So, if we are looking for a first-order try, we will use first order Green's functions, and so on and so forth. The guide to construct these trial solutions is to use the symmetries of the theory: Lorentz, gauge, etc. That is, we parametrize the ansätz via Feynman graphs with arbitrary internal topology, weight and mass, spanning the Poincaré-invariant space of the given theory.

It is known from mathematics that the Galerkin technique produces and approximate solution that converges to the exact solution as the number of terms goes to infinity. In practice, the series will have to be truncated and it is likely that, for nontrivial theories, calculations of the 8th-order will be very hard to compute. Thus, rapid convergence and high accuracy must be must be ensured (even with only a few terms in the ansätz). Once this convergence is in the mean (afterall, we are projecting the solution in the space of the u_i's), the accuracy will depend on the choice of inner product.

1.3 Dimensionally Deconstructing Quantum Field Theories (just to build them again)

Analyzing the dimensionally deconstructed ϕ^4 theory, (*a.k.a* ultra-local ϕ^4; 0 spacetime dimensions), a good choice of u_i's is given by [modified] Hermite polynomials (which carry an exponentially decreasing factor — Gaussian — in the inner product measure). Note that, from this 0-dimensional example, we are expected to obtain 3 solutions, given that the equation of motion is of the 3rd-order. However, using standard perturbative techniques, only one of these solutions can be obtained, namely that which is regular when $g\to 0$. The other two symmetry breaking solutions are not accessible via perturbation theory, making the use of non-perturbative techniques very desirable.

For higher dimensions, there is a "simple" generalization of the [modified] Hermite polynomials, given by:

$$H_n(\xi_1,\dots,\xi_n)=e^{1/2\,\xi^{\mathsf{T}}\xi}\,(-1)^n\,\frac{\partial^n}{\partial\xi_1\cdots\partial\xi_n}\,e^{-1/2\,\xi^{\mathsf{T}}\xi}\;;$$

where our n-th trial function is given by, $\mathcal{Z}^*=\sum_{i=1}^n c_i\,H_i$, where the c_i's are given by sums of Feynman graphs up to order n.

2. Mollifying Quantum Field Theory

2.1 Lattice Quantum Field Theory

Motivated by the question *"Is it possible to do Lorentzian Lattice QFT ($\exp\{i\,S[\phi]\}$)?"*, we engaged in the study of convolutions (used as filters) in order to try to smooth out the integrand above.

Note that, in Euclidian Lattice QFT, the object of study is,

$$\langle\mathcal{O}[\phi]\rangle=\int\mathcal{O}[\phi]\,\exp\{i\,S[\phi]\}\,\mathcal{D}\phi\mapsto\langle\mathcal{O}[\phi]\rangle_E=\int\mathcal{O}[\phi]\,\exp\{-S[\phi]\}\,\mathcal{D}\phi\;;$$

$$\langle\mathcal{O}[\varphi]\rangle_E^{\textsf{latt}}=\frac{1}{||\Phi||}\sum_{\varphi\in\Phi}\left\{\frac{1}{N^d}\sum_{i=1}^{N^d}\mathcal{O}[\varphi^{[i]}]\right\}\;;$$

where Φ is the set of all field configurations on the lattice and $\varphi^{[i]}$ are chosen with a probability density of $\exp\{-S_E[\varphi]\}$ (Markov Chains [ergodicity], Metropolis Monte Carlo).

The question of Lorentzian Lattice QFT is deeper than my appear at first sight. on the one hand, it would firstly enable the computation of highly oscillatory integrands including, but not limited to, phase transitions. Secondly, a single framework would be used for the treatment of all phases of a QFT. On the other hand, new techniques (analytic, numeric and algorithmic) have to be used in order to tackle this problem. Not to mention the fact that this *exercise* sheds some new light and understanding on old friends of ours, e.g., Feynman's Path Integral, Schwinger-Dyson's equation, phase transitions, etc.

2.2 Mollifiers or Approximate Identities

This technique is vastly used in mathematics in order to build [smooth] functions out of distributions (a.k.a generalized functions), using convolutions as "filters" and making wildly-behaving objects more tractable. The idea is to convolve two objects so that the result is a smooth function:

$$f_\epsilon(x)=\int_{\mathbb{R}}\eta_\epsilon(x-y)\,f(y)\,dy\;;$$

where ϵ is a parameter to control the approximation and $f_\epsilon(x)$ is the mollification of $f(x)$. The more frequently used mollifiers are,

Standard: $$\eta_\epsilon(x)=\begin{cases}C\,\exp\left\{\frac{1}{|x/\epsilon|^2-1}\right\} & ,|x|<1\;;\\ 0 & ,|x|\geqslant 1\;;\end{cases}$$

Gaussian: $$\eta_\epsilon(x)=\frac{1}{\sqrt{2\,\pi\,\epsilon^2}}\,\exp\left\{-\frac{1}{2}\,(x/\epsilon)^2\right\}\;;$$

Normalization: $$\int_{\mathbb{R}}\eta_\epsilon(x)=1\;.$$

2.3 Mollifying Quantum Field Theory

The idea is to use the mollification technology in order to smooth out (*i.e.*, filter) the highly oscillatory parts of the Path Integral in question, leaving only the most important contributions.

Mollified Generating Functional $$\mathcal{Z}[J]\mapsto\mathcal{Z}_\epsilon[J]\equiv\int_\Gamma\left\{\int_{\mathbb{R}}\eta_\epsilon[\phi-\varphi]\,e^{i\,S[\varphi,J]}\,\mathcal{D}\varphi\right\}\mathcal{D}\phi\;;$$

Fubini's Theorem $$\mathcal{Z}[J]=\mathcal{Z}_\epsilon[J]\;;$$

Property of Mollifiers $$\lim_{\epsilon\to 0}\mathcal{Z}_\epsilon[J]=\mathcal{Z}[J]\;.$$

Note that, by appropriately choosing Γ, you can pick out all the phase structure of the theory! Furthermore, this is a very nice way of introducing *block variables* straight into the Path Integral formulation of a QFT.

As a final point, note that you need to worry about non-local correlations when talking about Lattice QFT. A very novel approach is to use *Genetic Algorithms* in order to deal with this problem. (There are works that show that this is a better choice than the Swendsen-Wang and/or the Wolff algorithm.) This is a pioneer avenue pursued by us!

3. Topology Change in General Relativity

3.1 Chern-Simons Theory

Chern-Simons (CS) theory has a curious history: It was discovered in the context of anomalies in the 70's. It was only by the mid-80's that it was realized that ordinary Einstein gravity in $(2+1)$-dimensions is a natural example of a CS system. There is an intrinsic connection between CS and the [mathematical] theory of knots and link invariants. (Stablished by Witten more than 10 years ago: summer of 1988.) The table below shows some of the analogies:

Knot Theory	Chern-Simons Theory
knots and links	Wilson loops
polynomial invariants	vev's of products of Wilson loops
singular knots	operators of singular knots
Vassiliev invariants	coefficients of the perturbative series
configuration space integral	Landau gauge

The key to construct the CS form in 3-dim is as follows: the Pontryagin form, $\mathcal{P}=\mathrm{tr}(\mathcal{F}\wedge\mathcal{F})$, is closed, $\mathrm{d}\mathcal{P}=0$. ($\mathcal{F}=\mathrm{d}\mathcal{A}+\mathcal{A}\wedge\mathcal{A}$ is the curvature of the Lie algebra-valued connection 1-form $\mathcal{A}$, taken in the adjoint representation. Upon a gauge transformation, $\mathcal{F}\mapsto\mathcal{F}'=g^{-1}\,\mathcal{F}\,g$, where $g\in\mathfrak{g}$, the Lie algebra of the gauge group $\mathfrak{G}$. So using the cyclic property of the trace we see that the Pontryagin form, $\mathcal{P}$, remains invariant under gauge transformations.) By Poincaré's Lemma, $\mathcal{P}$ is locally exact, *i.e.*, $\mathcal{P}=\mathrm{d}\mathcal{Q}$. Thus, $\mathcal{Q}=L_{\textsf{SC}}$ is the CS Lagrangian, found to be $\mathcal{Q}=L_{\textsf{SC}}=\mathrm{tr}(\mathcal{A}\wedge\mathrm{d}\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A})$.

The essential ingredient in going to higher dimensions, n, is the existence of a closed $2\,n$-form, invariant under gauge transformations. It is straightforward to see that $\mathcal{Q}_{2\,n}=\mathrm{tr}(\mathcal{F}\wedge\cdots\wedge\mathcal{F})=\mathrm{tr}(\mathcal{F}^n)$ is what we are looking for.

More than sixty years of frustrated efforts to quantize this theory can explain the immediate attention drawn by Witten's observation that gravity in $(2+1)$-dim is an exactly solvable model! This means that the quantum theory can be completely and explicitly spelled out. This is due to the fact that $(2+1)$-dim gravity has no propagating degrees of freedom and, therefore, its quantum description is like that of a system of point particles. It is a particular case of a Topological [Quantum] Field Theory!

3.2 The Dirac Operator: Spectral Geometry, the Atyiah-Singer Index Theorem and [Non-]Commutative Geometry

With the help of the Dirac Operator — *i.e.*, a derivation of the spinor fields (defined on the Clifford [Geometric] Algebra of the bundle considered) — an analysis of the wave equation can be made, based on its eigenvalues and eigenvectors. That is, the idea is to tackle the question *Can one hear the shape of the drum?*, first posed by M. Kac (1966). This means to use the eigenvalues found above in order to reconstruct the manifold/bundle in which the wave equation is defined (boundary conditions are an essential ingredient of this potion). This subject is known as Spectral Geometry and, after the work of A. Connes, it has an intrinsic relation to [non-]commutative geometry, where a triple is given: $(\mathcal{A},\mathcal{H},D)$, where $\mathcal{A}$ is a C^*-algebra, $\mathcal{H}$ is the Hilbert space of square-integrable spinors and D is the Dirac Operator.

The Atyiah-Singer Index Theorem is essential in all of this construction, because it gives a tool that enables us to distinguish between different Dirac Operators and the topologies of the manifolds/bundles in which they are defined.

3.3 Morse Theory, Cobordisms, Topology Change and Phase Transitions

Given that the Chern-Simons form is a topological invariant, one can use it in order to measure whether a *topological phase transition* has occurred. The mathematical tools available for the analysis of topology change are, essentially, Morse Theory (generalization of the calculus of variations) and Cobordism Theory (relation between manifolds based on their boundaries). Furthermore, the study of Instantons/Solitons is highly based on the topological properties of the manifold/bundle where the particular QFT is defined.