The Effects of Spacetime on Yang-Mills Theories

or

Global (Topological) Aspects of YM Theories

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Outline:

1. Study classical YM using the Coulomb gauge;

2. Study the operator $\mathcal{P}$[§] perturbatively and non-perturbatively;

3. Spacetime manifold $\mathcal{M}$: Compact, Riemannian, without boundary, $\dim(\mathcal{M}) = 4 \Rightarrow$ discrete spectrum for the Laplacian;

4. $\mathcal{P}$: operator-valued function of the Laplacian. Obtain a Cauchy integral representation $\Rightarrow$ exact evaluation of its mean value on $\mathcal{L}^2(\mathcal{M})$[¶]; &

5. Residue's theorem $\Rightarrow$ mean value vanishes exactly.

[§]To be defined.

[¶]$\mathcal{L}^2(\mathcal{M}) \equiv \int_{\mathcal{M}} |\phi|^2 < \infty$; $\forall$ test function ϕ.

Connection with Cosmology

- Topology of Spacetime: global property of $\mathcal{M}$ — determines the form of the Laplace-Beltrami operator:

$$\Box_{\mathcal{M}} f(x_\mu) = \mathcal{D}_\mu[\partial^\mu f(x_\mu)] = \frac{1}{\sqrt{g}}\,\partial_\mu\left\{\sqrt{g}\,\partial^\mu\left[f(x_\mu)\right]\right\}$$

- *Robertson-Walker* metric

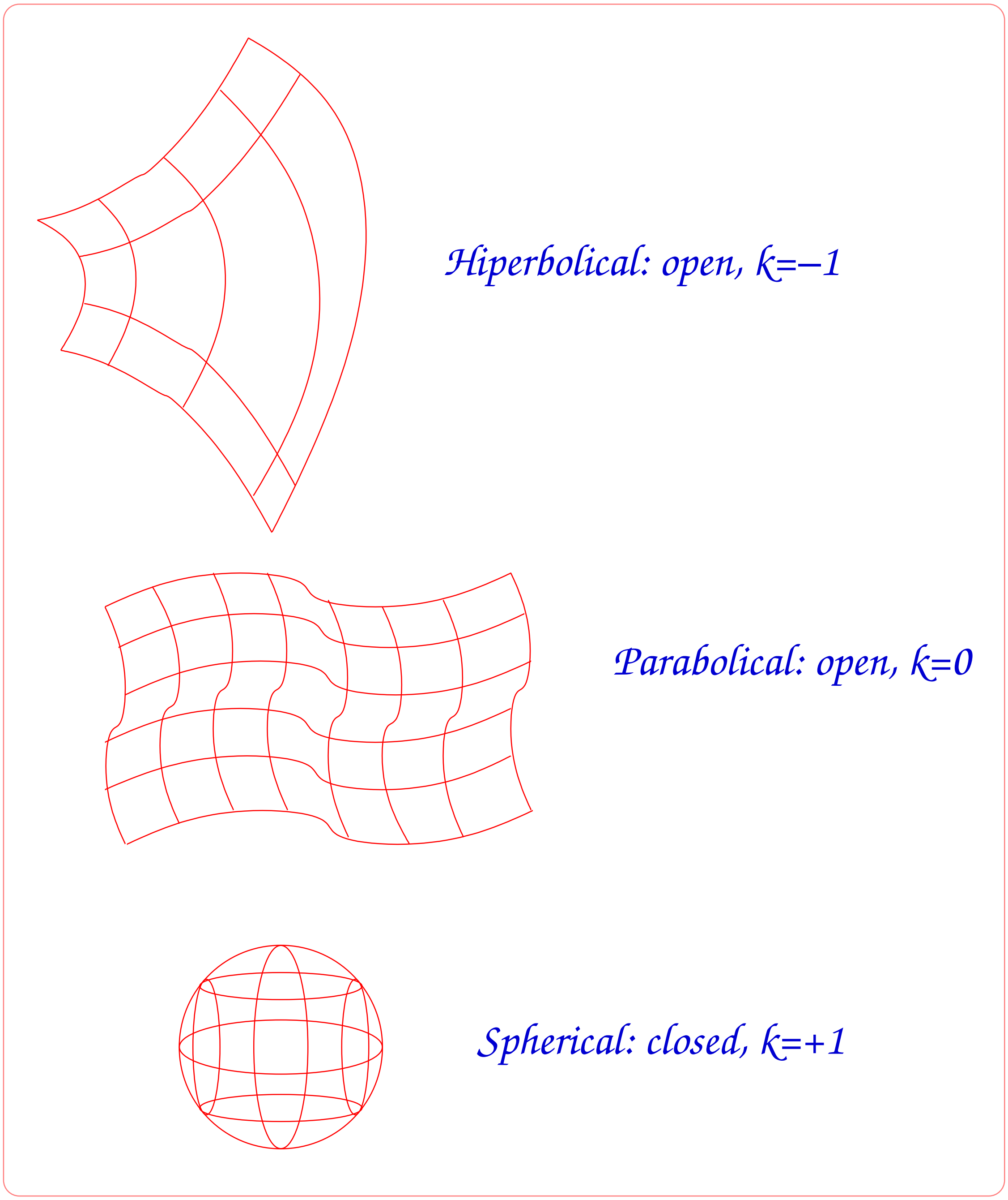
$$d\tau^2 = dt^2 - R^2(t)\left\{\frac{dr^2}{1-k\,r^2} + \underbrace{r^2\,d\theta^2 + r^2\,\sin^2(\theta)\,d\phi^2}_{=d\Omega^2}\right\}\,,$$

where $R(t)$ is known as: *cosmic scale factor*.

- $k=-1$: infinite/open space, with boundary, cannot be compact (hyperbolic space/geometry);

- $k=0$: infinite/open space, with boundary, cannot be compact (parabolic space/geometry, e.g. Minkowski); &

- $k = +1$: finite/closed space, can/cannot have boundaries, only one that can possibly be compact (spheric space/geometry).



- Much progress towards the theoretical understanding of gauge theories;

- Some unsolved problems remain, e.g., the *mass gap problem*;

- *Mass Gap* problem: prove that, $\forall$ compact, simple gauge group $\Rightarrow$ Quantum Yang-Mills on $\mathbb{R}^4$ has a Hamiltonian with no spectrum in the interval $(0, \delta)$, $\delta > 0$;

- Thus, suffices $\vdash$ $\mathcal{H}$, corresponding to the action $\mathcal{S} = -\frac{1}{4}\int_{\mathcal{M}} \mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$, has energy spectrum bounded from below, with a strictly positive lower bound, (δ).

“Why is it important?” ☺

1. Internal consistency of the formalism (Standard Model);
2. Short-range of the nuclear force;
3. Phenomenology of glueballs can be described by an action of the YM type; etc. . .

- The phenomenology of the Standard Model is described by YM $\Rightarrow$ study the problem in 4 dimensions.

- Consider the YM Lagrangian in the Coulomb gauge, with the *associated* decomposition into "electric" and "magnetic" parts. Coulomb gauge: non-covariant but, manifestly unitary.

Coulomb-Gauge Hamiltonian

YM Lagrangian $\Rightarrow$ expressed in terms of "electric" and "magnetic" fields:

$$\vec{\mathcal{E}}_i \equiv -\frac{d\vec{A}_i}{dt} - (\nabla_i + g\,\vec{A}_i \times)\vec{A}_0 = -\frac{d}{dt}\vec{A}_i + \frac{1}{g}\mathcal{D}_i\vec{\omega}$$

$$\varepsilon_{jki}\,\vec{\mathcal{B}}_i \equiv \nabla_j\vec{A}_k - \nabla_k\vec{A}_j + g\,\vec{A}_j \times \vec{A}_k$$

where:

$$\vec{A}_0 = -\frac{\vec{\omega}}{g}$$

$$\mathcal{D}_i = \nabla_i + g\,\vec{A}_i \times$$

$$\therefore \boxed{H = \frac{1}{2}\int \left(\vec{\mathcal{E}}_i \cdot \vec{\mathcal{E}}_i + \vec{\mathcal{B}}_i \cdot \vec{\mathcal{B}}_i\right)\, d^3x}$$

The notation is as follows: i, j, k are space indexes, $(\vec{\cdot})$ is a vector in the inner space and "$\times$" is the (iso)vector product (vector product in the inner space). We are using the summation convention and, the inner product in the inner space is given by the trace.

Let's use $\nabla_j \vec{A}_j = 0$ and decompose $\vec{\mathcal{E}}_i$:

$$\vec{\mathcal{E}}_i = \vec{\mathcal{E}}_i^{\perp} - \nabla_i \vec{\phi}$$

$$\vec{\mathcal{E}}_i^{\perp} = -\frac{d}{dt}\vec{A}_i + (\delta_{ij} - \nabla^{-2}\nabla_i \nabla_j)\,(\vec{A}_j \times \vec{\omega})$$

$$\therefore\ \nabla_i \vec{\mathcal{E}}_i^{\perp} = 0$$

$$\vec{\phi} = \vec{A}_0 + g\,\nabla^{-2}(\vec{A}_j \times \nabla_j \vec{A}_0)\ .$$

Since $\mathcal{D}_i \vec{\mathcal{E}}_i = 0$ (no sources) $\Rightarrow$

$$\vec{\Pi}_i^{\text{tr}} = -\vec{\mathcal{E}}_i^{\perp}$$

$$-\vec{A}_i \times \vec{\mathcal{E}}_i^{\perp} + \frac{1}{g}\nabla_i \mathcal{D}_i \vec{\phi} = 0$$

$$\Rightarrow\ \vec{\phi} = g\,(\nabla_k \mathcal{D}_k)^{-1}\,\vec{A}_i \times \vec{\mathcal{E}}_i^{\perp} = -g\,(\nabla_k \mathcal{D}_k)^{-1}\,\vec{A}_i \times \vec{\Pi}_i^{\text{tr}}\ .$$

$$\therefore\ \boxed{H = \frac{1}{2}\int \left((\vec{\mathcal{E}}_i^{\perp})^2 + (\nabla_i \vec{\phi})^2 + (\vec{\mathcal{B}}_i)^2\right)\,d^3x}$$

H: further simplified by means of our result for $\vec{\phi}$ and the identity and definition below:

$$(\nabla_i\vec{\phi})(\nabla_i\vec{\phi}) = \nabla_i(\vec{\phi}\nabla_i\vec{\phi}) + \vec{\phi}(-\nabla^2)\vec{\phi}$$

$$\vec{\sigma}_A \equiv \vec{A}_i \times \vec{\Pi}_i^{\perp}; \text{ charge carried by } \vec{A}_i$$

Classical Hamiltonian:

$$H = \frac{1}{2}\int\left[\left(\vec{\Pi}_i^{\perp}\right)^2 + \left(\vec{\mathcal{B}}_i\right)^2\right] + \\ + \frac{g^2}{2}\int \sigma_A^a(x)\,\mathcal{P}_{ab}(x,x')\,\sigma_A^b(x')\,d^3x d^3x'$$

where,

$$\mathcal{P}_{ij}^{ab} \equiv (\nabla_i\,\mathcal{D}_i^a)^{-1}\,(-\nabla^2)\,(\nabla_j\,\mathcal{D}_j^b)^{-1}$$

Convenient way of expressing $\nabla_i\mathcal{D}_i$: $V \equiv \nabla_i(\vec{A}_i\times) = \vec{A}_i \times \nabla_i \;\Rightarrow\; \nabla_i\mathcal{D}_i = \nabla^2 + g\,V \;\Rightarrow$

$$\begin{aligned}(\nabla_i\,\mathcal{D}_i)^{-1} &= (\nabla^2 + g\,V)^{-1} \\ &= \nabla^{-2}(\mathbb{1} - g\,V\,\nabla^{-2} + (g\,V\,\nabla^{-2})^2 + O(g^3)) \\ &= \nabla^{-2} - g\,\nabla^{-2}\,V\,\nabla^{-2} + \\ &\quad + g^2\,\nabla^{-2}\,V\,\nabla^{-2}\,V\,\nabla^{-2} + O(g^3)\end{aligned}$$

$\therefore$ algorithm for $\mathcal{P}$:

$$\mathcal{P} = -\nabla^2 + 2\,g\,\nabla^{-2}\,V\,\nabla^{-2} - 3\,g^2\,\nabla^{-2}\,V\,\nabla^{-2}\,V\,\nabla^{-2} + \cdots$$
$$= -\nabla^2 + \sum_{k=1}^{\infty} (-1)^{k+1}\,(k+1)\,g^k\,\nabla^{-2}\,(V\,\nabla^{-2})^k\ .$$

In the Quantum Theory:

- un-renormalized coupling constant: g_0;
- Faddeev-Popov determinant: $\gamma = \det(\nabla_i \mathcal{D}_i)$; &
- Equal-time commutation relations:

$$\Big[\,A_i^a(t,x), A_j^b(t,y)\,\Big] = 0 = \Big[\,\Pi_i^{\perp\,a}(t,x), \Pi_i^{\perp\,b}(t,y)\,\Big]$$
$$\Big[\,A_j^a(t,x), \Pi_k^{\perp\,b}(t,y)\,\Big] = i\delta^{ab}(\delta_{jk} - \nabla^{-2}\nabla_j\nabla_k)\delta^3(x-y)$$

- Hamiltonian operator in the Coulomb gauge:

$$\hat{H} = \frac{1}{2}\int \Big[\Big(\gamma^{-1}\vec{\Pi}_i^{\perp}, \gamma\vec{\Pi}_i^{\perp}\Big) + \Big(\vec{\mathcal{B}}_i, \vec{\mathcal{B}}_i\Big)\Big] +$$
$$+ \frac{g_0^2}{2}\int [\gamma^{-1}\sigma^a(x)]\,\mathcal{P}^{ab}(x,x')\,[\gamma\sigma^b(x')]\,d^3x d^3x'\ .$$

Perturbative Structure of $\mathcal{P}$

- Spectrum of $\widehat{H}$: define the Laplacian to be $-\nabla \equiv \triangle = L$, bounded from below $(\triangle^{-1} \equiv L^{-1} = \Gamma) \Rightarrow$

$$\Rightarrow \mathcal{P} = \frac{1}{\triangle} + \sum_{k=1}^{\infty} (k+1)\, g^k \frac{1}{\triangle} \left(\frac{1}{\triangle}\right)^k .$$

- $\forall$ compact, Riemannian manifold $(\mathcal{M})$ without boundary $\Rightarrow$ discrete spectral resolution of $\triangle$: complete, orthonormal set of eigenvectors u_l, eigenvalues λ_l: $\triangle u_l = \lambda_l\, u_l$, $\lambda_l \geq \lambda_0 \in \mathbb{R}$, $\forall\, l \in \mathbb{N}$.

- Left/right inverse: $L\,\Gamma = \Gamma\, L = \mathbb{1} \Rightarrow$

$$\Gamma\,(L u_l) = \Gamma\,(\lambda_l\, u_l) = u_l$$
$$\therefore\ \Gamma\, u_l = \frac{1}{\lambda_l}\, u_l$$

and

$$\Gamma^k\, u_l = \left(\frac{1}{\lambda_l}\right)^k u_l$$

$$\forall\,\phi \in \mathcal{L}^2(\mathcal{M}) \equiv \left\{ \phi : \mathbb{E} \to \mathcal{M} \,\middle|\, \int_{\mathcal{M}} |\phi|^2 < \infty \right\},$$

where $\mathbb{E}$ can be $\mathbb{R}^4$, $\mathscr{M}$ (Minkowski space), etc...
Consider two cases:

1. V is a constant, (idealization):
   Let c_l be the Fourier coefficients: $c_l = (\phi, u_l)$, then

$$\mathcal{P}\,\phi = \sum_{l=1}^{\infty} \frac{c_l}{\lambda_l} \left[1 + \sum_{k=1}^{\infty} (k+1) \left(\frac{g\,V}{\lambda_l} \right)^k \right] u_l \ .$$

Redefining

$$\omega_l \equiv \frac{g\,V}{\lambda_l}$$

$$\Rightarrow\ |\omega_l| < 1$$

$$\Rightarrow\ \mathcal{P}\,\phi = \sum_{l=1}^{\infty} \frac{c_l}{g\,V} \frac{\omega_l}{(1-\omega_l)^2}\, u_l$$

$$\therefore\ \forall\ |\omega_l| \in (0,1)\ \Rightarrow$$

$$(\phi, \mathcal{P}\,\phi) = \sum_{l,m=1}^{\infty} \frac{c_m^*\, c_l}{\lambda_l} \cdot \overbrace{\left(u_m, \frac{u_l}{(1-\omega_l)^2} \right)}^{\{u_l\}:\ \text{forms an orthonormal set}}$$

$$= \sum_{l=1}^{\infty} \frac{|c_l|^2}{\lambda_l\,(1-\omega_l)^2}\,.$$

2. $V \in \mathscr{C}^{\infty}$, (smooth function of position):

More careful handling $\Rightarrow$ basically the same result as above plus a *correction* given by an infinite sum involving commutators of the integral operator Γ with V (Baker-Campbell-Hausdorff formula), e.g.:

$$(V\,\Gamma)^2\,u_l = V^2\,\Gamma^2\,u_l + \frac{V}{\lambda_l}\,[\,\Gamma, V\,]\,u_l\;.$$

Exact, non-perturbative, analysis of $\mathcal{P}$

- g is not small, $\mathcal{P}$ is defined (as before) by

$$\mathcal{P} \equiv (\triangle - g\,V)^{-1}\,\triangle\,(\triangle - g\,V)^{-1}\ . \tag{1}$$

- Analysis for arbitrary g: desirable once our previous result leads to cumbersome formulæ $\Rightarrow$ no conclusions can be drawn.

- $\mathcal{P}$: operator-valued function of $\triangle$ $\Rightarrow$ $\mathcal{P} = f(\triangle)$ $\Rightarrow$ Cauchy integral formula (definition of the above):

$$\begin{aligned}\mathcal{P} = f(\triangle) &= \frac{1}{2\pi\,i}\oint_{\alpha}\frac{f(\mu)}{(\triangle - \mu\,\mathbb{1})}\,d\mu \\ &= \frac{1}{2\pi\,i}\oint_{\alpha}\frac{\mu}{(\mu - g\,V)^2}\,\frac{1}{(\triangle - \mu\,\mathbb{1})}\,d\mu\ ,\end{aligned}$$

  where α is a curve (smooth) in the $\mathbb{C}$ (Argand-)plane that goes around the discrete spectrum of $\triangle$ (bounded from below).

- $\forall\, \phi \in \mathcal{L}^2(\mathcal{M})$:

$$(\phi, \mathcal{P}\phi) = \sum_{l,r=1}^{\infty} \frac{c_r^*\, c_l}{2\pi\, i} \left(u_r, \left[\oint_\alpha \frac{\mu}{(\mu - g\, V)^2} \frac{1}{(\triangle - \mu\, \mathbb{1})}\, d\mu \right] u_l \right) .$$

- Study the action of the resolvent, $(\triangle - \mu \mathbb{1})^{-1}$, on the eigenvalues of the Laplacian:

$$\begin{aligned} \overbrace{(\triangle - \mu \mathbb{1})^{-1} (\triangle - \mu \mathbb{1})}^{=\mathbb{1}}\, u_l &= u_l \\ &= (\lambda_l - \mu)\, (\triangle - \mu \mathbb{1})^{-1}\, u_l \end{aligned}$$

$$\therefore\, (\triangle - \mu \mathbb{1})^{-1}\, u_l = (\lambda_l - \mu)^{-1}\, u_l$$

$$\Rightarrow \;\; (\phi, \mathcal{P}\phi) = \sum_{l,r=1}^{\infty} \frac{c_r^*\, c_l}{2\pi\, i}\, (u_r, \chi\, u_l)$$

$$\begin{aligned} \chi &\equiv \oint_\alpha h(\mu)\, d\mu \\ h(\mu) &\equiv \frac{\mu}{(\mu - g\, V)^2\, (\lambda_l - \mu)} . \end{aligned}$$

- Poles: $h(\mu)$ has one 2nd-order pole $\mu_1 = g\,V$ and one 1st-order pole $\mu_2 = \lambda_l \;\Rightarrow$ Residue's Theorem:

$$\chi = 2\pi\, i\, \left[\text{Res}(h(\mu))|_{\mu=gV} + \text{Res}(h(\mu))|_{\mu=\lambda_l}\right]$$

$$= 2\pi\, i\left[\lim_{\mu\to gV}\frac{d}{d\mu}\left((\mu-gV)^2\,\frac{\mu}{(\mu-gV)^2\,(\lambda_l-\mu)}\right)+\right.$$
$$\left. + \lim_{\mu\to\lambda_l}\left((\mu-\lambda_l)\,\frac{\mu}{(\mu-gV)^2\,(\lambda_l-\mu)}\right)\right]$$

$$= 2\pi\, i\left[\frac{1}{(\lambda_l-gV)}+\frac{gV}{(\lambda_l-gV)^2}-\frac{\lambda_l}{(\lambda_l-gV)^2}\right]$$

$\therefore\;\; \chi \equiv 0\;.$

- Thus, $(\phi, \mathcal{P}\phi) \equiv 0$, $\forall\, g$, for $\mathcal{M}$ compact and with no boundary!

Comments

- Original contribution: use of the discrete spectral resolution and the resolvent of the Laplacian; &

- Exact, non-perturbative result: $(\phi, \mathcal{P}\,\phi) = 0$, $\forall\, \phi \in \mathcal{L}^2(\mathcal{M})$.

Problems

1. Change the compact, 4-d Riemannian manifold, without boundary by a Lorentzian 4-manifold (e.g.: Minkowski) $\Rightarrow$ "decompactification limit"; &

2. Make the gauge group's role manifest.

References

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