BOUNDARY CONDITIONS FOR SCHWINGER–DYSON EQUATIONS AND VACUUM SELECTION

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INTRODUCTION

When attempting to solve a field theory or matrix model by using Schwinger–Dyson equations, one must address the problem that these equations do not possess a unique solution. This problem came to our attention when trying to numerically solve the equations for certain lattice field theories using a technique known as Source Galerkin [1], which is discussed elsewhere in these proceedings. It was found that to make the numerical method stable, one had to find some way of selecting the right boundary conditions. This led us to the more general question of how the boundary conditions are related to the phase diagram of a field theory or a matrix model. We shall summarize our work on this relation in this paper. Most of the details will appear elsewhere.

The naive resolution of the problem of how to select the boundary conditions is to simply pick the solution which corresponds to the path integral over real fields. However, within certain phases of many theories, it can be shown that the path integral solution is actually not the physical one. Furthermore in some matrix models the integral over real eigenvalues is not even convergent because of negative couplings. This forces the consideration of "exotic" solutions of the Schwinger–Dyson equations which have integral representations involving sums of integrals of the fields over various inequivalent complex contours. For theories with a local order parameter, symmetry breaking solutions are generated naturally by choosing a symmetry breaking set of contours. In the conventional approach to obtaining the broken phase, the real contour is chosen but a small symmetry breaking term is added to the action. This term is removed only after taking a thermodynamic limit, in which the number of degrees of freedom becomes infinite. In fact one can also obtain the broken phase by choosing a symmetry breaking boundary condition (contour) and then taking the thermodynamic limit directly. This is a simple example showing that the exotic solutions are not necessarily unphysical. We conjecture that this is true even for theories with a nonlocal order parameter, though this has yet to be demonstrated.

The difficulty in choosing the correct boundary conditions lies in the fact that there are so many of them. Furthermore since the Schwinger–Dyson equations satisfied by the partition function are linear, there naively appears to be a continuum of mixed phases, which does not make physical sense. Most of this problem is resolved by taking the thermodynamic limit. In this limit the solutions associated with many boundary conditions coalesce, while other boundary conditions do not lead to solutions with a thermodynamic limit This often leaves a countable number of discrete solutions, although sometimes a continuum of distinct solutions survives, yielding something analogous to the theta vacua in Yang Mills theories. The set of solutions which survives in the thermodynamic limit may include both the physical vacua, and false vacua with a complex free energy. It can be shown that the set of boundary conditions for which a thermodynamic limit exists varies along paths in the space of coupling constants. This set may vary smoothly or it may be discontinuous at certain points along the path. We have found it easiest to study the behavior of this set in the context of one matrix models, while field theories have proven less tractable. However we suspect that our conclusions are quite general. One important conclusion is that the set of boundary conditions with a thermodynamic limit changes discontinuously as one crosses a phase boundary.

BOUNDARY CONDITIONS IN ZERO DIMENSION

To illustrate the multiple solutions of the Schwinger–Dyson equations we begin with the simple example of a zero dimensional theory with the polynomial action $S(\phi) = \frac{1}{n} g_n \phi^n$. The Generating functional of disconnected green's functions $Z(J)$ satisfies the Schwinger–Dyson equation

$$(g_n \frac{\partial}{\partial J^{n-1}} - J) Z(J) = 0 \tag{1}$$

The order of this equation is determined by the highest order term in the polynomial action. If the highest order term is ϕ^k then there is an $k-1$ parameter class of solutions. One of these parameters is just the overall normalization of Z, so there is a $k-2$ parameter class of solutions with distinct green's functions. An integral representation of the solutions is

$$Z(J) = \int_\Gamma d\phi e^{-S(\phi)+J\phi} \tag{2}$$

It is easy to show that this satisfies the Schwinger–Dyson equation provided that

$$e^{-S(\phi)+J\phi}\Big|_{\partial\Gamma} = 0 \tag{3}$$

This is true if $ReS(\phi) \to +\infty$ asymptotically on the contour Γ. For the polynomial action this condition becomes $Reg_k\phi^k \to +\infty$. Therefore there are k wedge shaped domains in the complex plane in which the contour can run off to infinity. There are $k-1$ independent contours satisfying (3), which is consistent with the order of the Dyson Schwinger equation. In general it is the behavior of the action for large fields which controls the solution set, even for many degrees of freedom. The condition (3) can be used to construct the solution set even when the action is non polynomial and the order of the Schwinger–Dyson equations is unclear. For example the action $S = \beta cos\phi$, (one plaquette QED), can be shown by this method to yield a two parameter class of solutions for Z. A basis set of solutions is given by the contours

$$\Gamma_1 = [-i\infty, +i\infty] \tag{4}$$

and

$$\Gamma_2 = [-i\infty, 0] + [0, 2\pi] + [2\pi, 2\pi + i\infty] \tag{5}$$

The difference between these two solutions is the usual solution in which the contour runs from 0 to 2π. Note that in general the exotic solutions for a gauge theory correspond to a complexification of the gauge group. In this simple case it is possible to verify the counting of solutions by coupling sources J and $\bar{J}$ to the loop variables $e^{i\phi}$ and $e^{-i\phi}$. The Schwinger–Dyson equation is then,

$$[\beta(\partial_J - \partial_{\bar{J}}) - 2(J\partial_J - \bar{J}\partial_{\bar{J}})]Z(J, \bar{J}) = 0 \tag{6}$$

which when combined with the constraint,

$$\frac{\partial}{\partial J}\frac{\partial}{\partial \bar{J}} Z = Z \tag{7}$$

yields a two parameter class of solutions for Z.

BOUNDARY CONDITIONS IN THE GENERAL CASE

Let us consider the solution set for a lattice field theory. For theories in which the large field behavior of the action is dominated by a local term, such as $g_k\phi^k$, the construction of the solution set is a simple generalization of the zero dimensional construction. One simply chooses one of the zero dimensional contours for each field at each lattice site. An arbitrary solution is obtained by summing solutions with a definite set of contours. The solution set is then somewhat reduced by imposing the lattice symmetries, but is still very large. In an NxN matrix model, the lattice site label is replaced by an eigenvalue label, and since the interaction between eigenvalues is only logarithmic, the highest order term in the potential determines the allowable contours for each eigenvalue. For theories in which the large field behavior of the action is not dominated by a local term, the construction of the solution set is somewhat more complicated. This appears to be the situation in general for lattice theories with a non-local order parameter.

For a generic action with a finite number of degrees of freedom The number of independent solutions of the Schwinger–Dyson equations is exactly equal to the number of classical solutions, including the complex solutions. The exception to this rule occurs when the action has flat directions, or extrema with vanishing second derivative, in which case every term in the perturbative expansion about the classical solution diverges. For example in zero dimensions, the potential $V = g\phi^4$ has only one classical solution, but there are three independent solutions of the Dyson Schwinger equations. Assuming that we are not considering such an exceptional case, the borel resummation of the perturbation series about any classical solution can be shown to yield one or more exact solutions of the Schwinger–Dyson equations with some exotic integral representation. One can see to which contours these solutions must correspond by taking the weak coupling limit. The contours, assumed to be of constant phase, must avoid all the classical solutions of equal or lower action than the one whose perturbative expansion is being borel resummed. There are often many choices of such contours which correspond to the various choices for avoiding positive real singularities in the borel variable. Thus the complete set of solutions of the classical equations yields an overcomplete set of solutions of the Schwinger–Dyson equations. An arbitrary solution is obtained by taking Linear combinations of these solutions,

Thus away from the thermodynamic limit, there are a continuum of phases, resembling theta vacua. To complete the classification of these phases one needs to give a rule for identifying solutions in the same phase at different values of the coupling constants. More precisely, one needs some set of first order differential equations in the coupling constants, which the partition function in a given phase should satisfy. A natural choice is given by the Schwinger action principle, which may be stated as,

$$(\frac{\partial}{\partial g} - < \frac{\partial S}{\partial g} >) Z = 0 \tag{8}$$

or, for the example of the zero dimensional polynomial action, as

$$(\frac{\partial}{\partial g_n} - \frac{1}{n}\frac{\partial}{\partial J^n}) Z = 0 \tag{9}$$

For a solution satisfying the action principle, it is possible to hold the associated measure, or sum over contours, fixed locally as one moves about the space of coupling constants. Note however that the action principle does not allow one to fix the measure globally if there is a branch cut in any of the couplings. For instance in the zero dimensional ϕ^4 theory, if one rotates the phase of the coupling constant by 2π in accordance with the action principle, the contour of integration rotates by $\frac{\pi}{2}$ yielding an inequivalent solution. One could have chosen another set of first order differential equations instead of the action principle. However the Schwinger Action operators which annihilate the partition function commute with the Schwinger–Dyson operators which annihilate the partition function. If in the thermodynamic limit a continuum of solutions to the Schwinger–Dyson equations reduces to a discrete set, then in general the action principle turns out to be automatically satisfied. For instance in a matrix model the action principle is satisfied order by order in the $\frac{1}{N}$ expansion without having to be imposed by hand.

VACUUM SELECTION

Having classified the solutions far from the thermodynamic limit, one needs some way of reducing the solution set. It is tempting to invoke such requirements as reality and positivity. Reality however is not much of a constraint, because if all the couplings of the theory are real, one can always take linear combinations of exotic solutions such that all Green's functions become real. Thus this is not so strong a constraint and also throws away false vacuum solutions which may be of physical interest. One might also be tempted to invoke positivity, since only the integral over real fields is obviously positive. However it is not in general necessary to throw out members of the solution set by hand and it is dangerous to invoke reality and positivity before taking a thermodynamic limit. The thermodynamic limit alone does most of the job of reducing the solution set. This can seen very explicitly in matrix models. Consider a model of an N x N hermitian matrix M,

$$Z = \int dM e^{-NtrV(M)} \tag{10}$$

M may be written as $U\Lambda U^\dagger$ where Λ is diagonal, and then if the integral over U is performed one gets,

$$Z = \int \prod_n d\Lambda_n \Delta^2[\Lambda] e^{-N\sum_n V(\Lambda_n)} \tag{11}$$

where $\Delta[\Lambda]$ is the Vandermonde determinant

$$\Delta[\Lambda] = \prod_{n<m} (\Lambda_n - \Lambda_m) \tag{12}$$

One can separately choose a contour for each eigenvalue by the condition $ReV(\Lambda_n) \to +\infty$ for large Λ_n. The boundary condition problem is then most easily understood if one solves the model by the method of orthogonal polynomials. The reader is referred elsewhere for details [2]. To summarize the method very briefly, polynomials $P_n(\lambda) = \lambda^n + ...$ are defined such that,

$$\int d\lambda e^{-NV(\lambda)} P_n(\lambda) P_m(\lambda) = h_n \delta_{n,m} \tag{13}$$

where the integral over λ is over some permissible complex contour or sum of complex contours. The P_n have the property that

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + S_n P_n(\lambda) + R_n P_{n-1}(\lambda) \tag{14}$$

The partition function and the Green's functions may be written in terms of the coefficients R_n and S_n These coefficients satisfy a set of recursion relations, and a smooth thermodynamic (large N planar) limit is obtained if they may be written, in this limit, in terms of functions of $x = \frac{n}{N}$ on the interval $[0, 1]$. These functions are easily obtained from the recursion relations. One can then look for the possible contours which give such limiting functions, using the fact the R_n and S_n coefficients are related to the green's functions of the zero dimensional theory with the action $NV(\lambda)$ as $N \to \infty$. In this way it is easy to see that the solutions associated with a very large number of boundary conditions either coalesce or have no thermodynamic limit as $N \to \infty$ At certain critical values of the couplings, such as the points about which a double scaling limit may be taken, the functions of x which one obtains in the planar limit become complex within the interval [0,1]. This means that as one crosses these critical domains, the boundary conditions which give real solutions no longer have a thermodynamic limit. A change in the boundary conditions which lead to a thermodynamic limit is a general phenomenon which occurs as one crosses a phase boundary.

As an interesting aside, we note that there are some very exotic boundary conditions in matrix models which do have a thermodynamic limit. It has been shown that in the planar limit matrix models with degenerate minima have a continuum of solutions analogous to theta vacua [3]. One can find integral measures which account for some of these solutions [4]. However there are too many such solutions to be accounted for by integral measures which are the same for every eigenvalue. Furthermore these solutions persist even when a small term is added to the action lifting the degeneracy. Yet if one removes the degeneracy but restricts oneself to solutions with the same integral measure at each eigenvalue, then the associated $S(0)$ and $R(0)$ can only take certain discrete values. This can be seen from the fact that $S(0)$ and $R(0)$ are the one point and two point connected green's functions of the zero dimensional theory with action $NV(\lambda)$ as $N \to \infty$. A discrete set of values for $S(0)$ and $R(0)$ would not permit a continuum of theta vacua. Consequently the theta vacua can only arise from boundary conditions for which the measure is not a simple product of the same measure at each eigenvalue. One could instead take sums of solutions with staggered boundary conditions, in which the contours are different for different eigenvalues.

There are several phenomena which occur in the thermodynamic limit which appear to be disparate. One is the collapse of the space of boundary conditions, due to the fact that many boundary conditions do not lead to thermodynamic limit. The other is the appearance of new nonanalyticity and phase boundaries in the coupling constants due to the accumulation of Lee-Yang zeroes [5]. In fact these are not really disparate. We will give a heuristic argument below. The collapse of the space of boundary conditions means that as one varies the couplings, one must also vary the boundary conditions (contours), since the set of boundary conditions with a thermodynamic limit depends on where one is in the space of coupling constants. It is then possible that by following a closed path in coupling constant space, one does not return to the boundary condition with which one started. This would mean that the thermodynamic limit has introduced a nonanalyticity in the coupling constants, which in turn requires an accumulation of Lee-Yang zeroes. Note that there are simple zero dimensional analogues of this. Consider the polynomial action $S = \sum_{n=1}^{k} g_n \phi^n$. The solutions are analytic in in all but the highest coupling constant g_k. If one follows a closed path in the complex plane of any of the lower coupling constants, one does not have to change the contour to maintain the convergence, and thus there is only a single Riemann sheet in the lower coupling constants. If however one tunes g_k to zero, then the space of boundary conditions shrinks. If one now rotates the phase of g_{k-1} by a large amount, then one must also rotate the contour of integration, whereas the contour could be held fixed for non-zero g_k. Green's functions become multiply sheeted in g_{k-1}

The conventional tool for studying the phase structure of theories with a local order parameter is the effective potential. It is interesting to note the consequences of our analysis of of boundary conditions on the form of the effective potential. In fact, far from the thermodynamic limit there are many effective potentials. Consider again the zero dimensional example. The function $\phi(J) = \frac{\partial}{\partial J} ln Z(J)$, satisfies a nonlinear differential Dyson Schwinger equation of order $k-2$. The effective potential $\Gamma(\phi)$ is defined by the relation $J = \frac{d\Gamma}{d\phi}$. Any effective potential can only correspond to a discrete number of solutions $\phi(J)$ out of the full $k-2$ parameter class. In fact in zero dimensions there is no non-analyticity in J so their is only one solution associated with each effective potential. Note that order by order in a loop expansion, $\phi(J)$ is multiply sheeted and the effective potential appears to possess multiple minima which correspond to different boundary conditions. This is a spurious feature of the loop expansion in zero dimensions. Actually $\phi(J)$ has poles in J rather than branch cuts. In a thermodynamic limit these poles can coalesce, in which case several boundary conditions become described by the same exact effective potential. Many effective potentials associated with different boundary conditions coalesce in the thermodynamic limit, while effective potentials associated with other boundary conditions do not have a well defined thermodynamic limit.

CONCLUSION

The phase structure of field theories and matrix models appears to have a very natural interpretation in terms of the boundary conditions of Schwinger–Dyson equations. There are several interesting open questions. One question concerns the relation between the various boundary conditions and the phases of a theory with a nonlocal order parameter. As yet we have not been able to construct such a relation. Another interesting question is whether by using exotic contours one could formulate a practical method to study theta vacua on the lattice.

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